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# Reverse Mathematics and partial orders

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To my parents

# Abstract

We investigate the reverse mathematics of several theorems about partial orders. We mainly focus on the analysis of *scattered* (no copy of the rationals) and *FAC* (no infinite antichains) partial orders, for which we consider many characterization theorems (for instance the well-known Hausdorff's theorem for scattered linear orders).

We settle the proof-theoretic strength of most of these theorems. If not, we provide positive and negative bounds (for instance showing that the statement is provable in $\mathsf{WKL}_0$ but not in $\mathsf{WWKL}_0$).

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# Contents

# 1 Introduction

## 1.1 Background

The effective content of partial orders has been extensively studied in computability theory (see [Spe55, Har68, Ros82, Roy90, Dow98, Her01, Mon07, CDSS12]) and reverse mathematics (see [Clo89, FH90, Mar93, Hir94, DHLS03, CMS04, Mar05, Mon06, MM09, MS11]). We focus on the reverse mathematics of partial orders, in particular *scattered* and *FAC* partial orders.

A good reference for order theory is Fräissé's monograph [Fra00]. The main reference for reverse mathematics is Simpson's book [Sim09].

The goal of reverse mathematics is to measure the proof-theoretical strength of mathematical theorems by classifying which set-existence axioms are needed to establish their proofs. In practice, we work with fragments, or subsystems, of second-order arithmetic, finding the weakest system $S$ that suffices to prove a given theorem $\tau$: this means that $S$ proves $\tau$ and all the axioms of $S$ are provable from $\tau$ over a weaker system $S_0$. It is worth noticing that second-order arithmetic allows only the study of statements about countable (or countably coded) objects. Therefore, most set-theoretic techniques cannot be used in this context, and we often need to find a new proof when the classical proof heavily relies on stronger set existence axioms.

**Historical note.** Second-order arithmetic was first introduced and developed by Hilbert and Bernays [HB44a, HB44b]. In the sixties Harvey Friedman [Fri67, Fri69, Fri70, Fri71a, Fri71b] began a metamathematical investigation of subsystems of second-order arithmetic aiming to show the necessary use of strong set-theoretic assumptions in ordinary mathematics. Thereafter Friedman [Fri75, Fri76] initiated the program of reverse mathematics in order to address the following question:

Which are the "proper axioms" to prove theorems in mathematics?

Later on Friedman, Simpson and many others pursued the theme of reverse mathematics bringing forth several works (see Simpson's book [Sim09] for an accurate bibliography). Nowadays reverse mathematics is a lively and active field of research in the area of computability theory and proof theory.

## 1.2 Outline

In Chapter 1, we introduce reverse mathematics and the main systems of second-order arithmetic. We point up a few standard reverse mathematics and computability-theoretic results which will be used throughout the thesis. Then we give the main definitions for partial orders and fix notation and conventions.

In Chapter 2, we analyze a structure theorem due to Bonnet which characterizes FAC partial orders in terms of initial intervals. We also consider a theorem due to Erdös and Tarski about strong antichains. It turns out that (one direction of) Bonnet theorem and Erdös-Tarski theorem are equivalent from the viewpoint of reverse mathematics: in fact they are both equivalent to $\mathsf{ACA}_0$ (we notice that the classical proof of Bonnet theorem makes use of Erdös-Tarski result). For the other direction of Bonnet theorem we provide a partial result showing that such implication lies below $\mathsf{WKL}_0$ and strictly above $\mathsf{RCA}_0$.

In Chapter 3, we consider four classically equivalent definitions of scattered FAC partial orders and provide a reverse mathematics analysis similar to that for well-partial orders given in [CMS04]. The analysis leads us to consider a partition theorem on the rationals due to Erdös and Rado. On the side, we also improve some results of [CMS04].

In Chapter 4, we study another theorem by Bonnet which gives a characterization of scattered FAC partial orders. This theorem says that a countable partial order is scattered and FAC if and only if there are countably many initial intervals. We show that one direction (left to right) is equivalent to $\mathsf{ATR}_0$ while the other is provable in $\mathsf{WKL}_0$, but not in $\mathsf{RCA}_0$. Once again, we are not able to settle the exact reverse mathematics strength of the latter statement, which turns out to be an interesting problem from the viewpoint of reverse mathematics.

In Chapter 5, we consider several results about scattered linear orders due to Hausdorff. In particular (Section 5.5), we analyze Hausdorff's classification theorem for scattered linear orders and prove its equivalence with $\mathsf{ATR}_0$.

In Chapter 6, we consider two classification theorems which are the analogue of Hausdorff's theorem for scattered linear orders with respect to the class of scattered FAC partial orders and the class of countable FAC partial orders respectively. In either case we provide a proof in $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$ for the hard direction and a proof in $\mathsf{ACA}_0$ for the easy one.

In Chapter 7, we study the relation between partial orders and their linear extensions introducing the notion of linearizability. We then consider the statement "$\tau$ is linearizable" for the order types $\omega, \omega^*, \omega + \omega^*$ and $\zeta$ and obtain equivalences with $\mathsf{B}\mathbf{\Sigma}^0_2$ and $\mathsf{ACA}_0$.

## 1.3 Reverse Mathematics

The language of second-order arithmetic has symbols $0, 1, +, \cdot, <$, set membership $\in$, and two types of variables: number variables $n, m, \ldots$ for the natural numbers and set variables $X, Y, \ldots$ for sets of natural numbers. Generally, we use $\omega$ to mean the standard natural numbers, while we define $\mathbb{N}$ by the formula $(\forall n)(n \in X)$.

We define a hierarchy of formulas by starting with $\mathbf{\Sigma}^0_0$ ($\mathbf{\Pi}^0_0$) formulas, which are the ones with only bounded quantifiers $\exists n < m$ and $\forall n < m$. We then define inductively $\mathbf{\Sigma}^0_{n+1}$ ($\mathbf{\Pi}^0_{n+1}$) formulas $\exists n\varphi$ ($\forall n\varphi$) where $\varphi$ is $\mathbf{\Pi}^0_n$ ($\mathbf{\Sigma}^0_n$). The formulas so defined are called arithmetical. We extend the hierarchy by defining $\mathbf{\Sigma}^1_n$ ($\mathbf{\Pi}^1_n$) formulas. The arithmetical formulas are $\mathbf{\Sigma}^1_0$ ($\mathbf{\Pi}^1_0$). A formula $\exists X\varphi$ ($\forall X\varphi$) is $\mathbf{\Sigma}^1_{n+1}$ ($\mathbf{\Pi}^1_{n+1}$), where $\varphi$ is $\mathbf{\Pi}^1_n$ ($\mathbf{\Sigma}^1_n$).

A comprehension axiom for second order arithmetic is

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \varphi(n)),$$

where $\varphi$ is a formula not mentioning $X$. Basically, we are saying that the set of natural numbers satisfying the property $\varphi$ exists.

Comprehension for $\mathbf{\Pi}^i_n$ formulas ($i = 0, 1$) is defined by taking $\varphi$ over all the $\mathbf{\Pi}^i_n$ formulas. Comprehension for $\mathbf{\Delta}^i_n$ formulas is defined by

$$(\forall n)(\varphi(n) \Leftrightarrow \psi(n)) \implies (\exists X)(\forall n)(n \in X \Leftrightarrow \varphi(n)),$$

where $\varphi$ is any $\mathbf{\Sigma}^i_n$ formula and $\psi$ is any $\mathbf{\Pi}^i_n$ formula.

We briefly recall the main subsystems of second order arithmetic (known as "the big five"). In order of increasing strength, they are: $\mathsf{RCA}_0$, $\mathsf{WKL}_0$, $\mathsf{ACA}_0$, $\mathsf{ATR}_0$, and $\mathbf{\Pi}^1_1$-$\mathsf{CA}_0$. Each system contains the algebraic axioms of Peano Arithmetic (i.e. the axioms for $0, 1, +, \cdot, <$).

$\mathsf{RCA}_0$ is the usual base system over which we prove equivalences. We restrict comprehension to $\mathbf{\Delta}^0_1$ formulas and induction to $\mathbf{\Sigma}^0_1$ formulas. $\mathsf{RCA}_0$ roughly corresponds to computable or constructive mathematics. The next, $\mathsf{WKL}_0$, consists of $\mathsf{RCA}_0$ plus Weak König's lemma: "every infinite binary tree has an infinite path". $\mathsf{ACA}_0$, Arithmetical Comprehension, is the system obtained by allowing comprehension for arithmetical formulas. $\mathsf{ATR}_0$, Arithmetical Transfinite Recursion, allows iterations of arithmetical comprehension along any well-order. $\mathbf{\Pi}^1_1$-$\mathsf{CA}_0$ is obtained from $\mathsf{RCA}_0$ extending comprehension to $\mathbf{\Pi}^1_1$ formulas. We refer the reader to Simpson [Sim09] for a detailed description of second-order arithmetic.

We routinely use the following equivalences when proving our results.

**Theorem 1.3.1** ([Hir87]). *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{B}\mathbf{\Sigma}^0_2$ *($\mathbf{\Sigma}^0_2$ bounding principle): for every $\mathbf{\Sigma}^0_2$ formula $\varphi$,*
$(\forall i < n)(\exists m)\varphi(i, n, m) \implies (\exists k)(\forall i < n)(\exists m < k)\varphi(i, n, m)$*;*

(2) $\mathsf{RT}^1_{<\infty}$ *(Infinite Pigeonhole Principle): $(\forall n)(\forall f : \mathbb{N} \to n)(\exists A \subseteq \mathbb{N}$ infinite$)(\exists c < n)(\forall m \in A)(f(m) = c)$.*

**Theorem 1.3.2** ([Sim09], Lemma IV.4.4). *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{WKL}_0$*;*

(2) *$\mathbf{\Sigma}^0_1$ separation: for all $\mathbf{\Sigma}^0_1$ formulas $\varphi(n), \psi(n)$, if $(\forall n)\neg(\varphi(n) \wedge \psi(n))$, then there exists a set $Z$ such that*

$$(\forall n)(\varphi(n) \implies n \in Z) \text{ and } (\forall n)(\psi(n) \implies n \notin Z).$$

(3) *for all one-to-one (total) functions $f, g\colon \mathbb{N} \to \mathbb{N}$, if $(\forall n, m)(f(n) \neq g(m))$, then there exists a set $Z$ such that*

$$\operatorname{ran}(f) \subseteq Z \text{ and } Z \cap \operatorname{ran}(g) = \emptyset.$$

**Theorem 1.3.3** ([Sim09], Lemma III.1.3). *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{ACA}_0$*;*

(2) *$\mathbf{\Sigma}^0_1$ comprehension;*

(3) *for every one-to-one (total) function $f\colon \mathbb{N} \to \mathbb{N}$ the set $\{n\colon (\exists m) f(m) = n\}$ exists.*

The following is [Sim09, Theorem V.5.2].

**Theorem 1.3.4** ([Sim09]). *Over $\mathsf{RCA}_0$, the following are equivalent:*

(1) $\mathsf{ATR}_0$*;*

(2) *for any sequence of trees $\{T_i\colon i \in \mathbb{N}\}$ such that every $T_i$ has at most one path, the set $\{i \in \mathbb{N}\colon [T_i] \neq \emptyset\}$ exists.*

**Theorem 1.3.5** ([Sim09]). *The following are pairwise equivalent over* $\mathsf{ACA}_0$*:*

(1) $\mathsf{ATR}_0$*;*

(2) *if an analytic set $A$ is uncountable, then $A$ has a non-empty perfect subset;*

(3) *if a tree $T \subseteq 2^{<\mathbb{N}}$ has uncountably many paths, then $T$ has a non-empty perfect subtree;*

(4) *if a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has uncountably many paths, then $T$ has a non-empty perfect tree.*

## 1.4 Computability theory and $\omega$-models

The following basic facts will be used to establish a few unprovability results via $\omega$-models. In particular we show that some statements are not provable in $\mathsf{WKL}_0$ and $\mathsf{WWKL}_0$ (see subsection 3.6).

**Definition 1.4.1.** An *$\omega$-model* is a model for the language of second-order arithmetic of the form $(\omega, S)$, where $S \subseteq \mathcal{P}(\omega)$ and the interpretation of $0, 1, +, \cdot, <$ is standard.

We assume familiarity with the main concepts of computability theory (for an introduction see for instance [DH10, Chapter 2]). We mention that computability-theoretic results are often used to build $\omega$-models (by relativization and iteration) and separate one principle from another (for instant the Low Basis Theorem yields an $\omega$-model of $\mathsf{WKL}_0$ which is not a model of $\mathsf{ACA}_0$).

**Theorem 1.4.2** ([Sco62]). *For every set $X$ of Peano degree there exists a model $M$ of* $\mathsf{WKL}_0$ *such that $(\forall Y \in M) Y \leq_T X$.*

**Theorem 1.4.3** ([JS72]). *There is a low set of Peano degree.*

**Theorem 1.4.4** ([YS90]). *For every Martin-Löf random set $X$ there exists a model $M$ of* $\mathsf{WWKL}_0$ *such that* $(\forall Y \in M) Y \leq_T X$.

**Theorem 1.4.5** ([ML66]). *The class of Martin-Löf random reals has measure* $1$.

## 1.5 Terminology, notation and basic facts

All definitions in this section are made within $\mathsf{RCA}_0$.

### 1.5.1 Partial orders

A *partial order* is a pair $(P, \leq_P)$, where $P \subseteq \mathbb{N}$ and $\leq_P$ is a reflexive, antisymmetric and transitive binary relation on $P$. We usually refer to $(P, \leq_P)$ simply as $P$ and we use $\preceq$ or other symbols instead of $\leq_P$ when there is no danger of confusion.

- We say that $x, y \in P$ are *comparable* if $x \preceq y$ or $y \preceq x$. If $x$ and $y$ are incomparable we write $x \perp y$.
- If $x \in P$, we let $P(\perp x) = \{y \in P \colon x \perp y\}$ and define the *upper and lower cones* determined by $x$ setting
$$P(\succeq x) = \{y \in P \colon x \preceq y\} \text{ and } P(\preceq x) = \{y \in P \colon y \preceq x\}.$$
$P(\succ x)$ and $P(\prec x)$ are defined in the obvious way.
- If $X \subseteq P$ we write $\downarrow X$ for the *downward closure* of $X$, i.e. $\bigcup_{x \in X} P(\preceq x)$. Notice that the existence of $\downarrow X$ as a set is equivalent to $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$.
- A *linear order* (or a *chain*) is a partial order such that all elements are pairwise comparable.
- A subset $A \subseteq P$ is an *antichain* if all its elements are pairwise incomparable, i.e. $(\forall x, y \in A)(x \neq y \implies x \npreceq y \wedge y \npreceq x)$.
- A partial order is called *FAC* (for *finite antichain condition*) if it does not contain infinite antichains.

- A linear order $P$ is *dense* if for all $x, y \in P$ such that $x \prec y$ there exists $z \in P$ with $x \prec z \prec y$.

- We say that $x, y \in P$ are *compatible in* $P$ if there is $z \in P$ such that $x \preceq z$ and $y \preceq z$. A subset $S \subseteq P$ is a *strong antichain in* $P$ if its elements are pairwise incompatible in $P$, i.e. $(\forall x, y \in S)(\forall z \in P)(x, y \preceq z \implies x = y)$.

- A subset $I \subseteq P$ is an *initial interval* of $P$ if $(\forall x, y \in P)(x \preceq y \wedge y \in I \implies x \in I)$. An initial interval $A$ of $P$ is an *ideal* if every two elements of $A$ are compatible in $A$, i.e. $(\forall x, y \in A)(\exists z \in A)(x \preceq z \wedge y \preceq z)$.

- $P$ is *well-founded* if it contains no infinite descending sequences. By an infinite descending sequence we mean a function $f \colon \mathbb{N} \to P$ such that $f(i) \succ f(j)$ for all $i < j$.

- A *well-order* is a well-founded linear order. We use set-theoretic notation and denote well-orders by $\alpha, \beta, \gamma \ldots$. We write $\beta < \alpha$ to mean that $\beta$ is an element of $\alpha$;

- $P$ is said to be a *well-partial order* if for every function $f \colon \mathbb{N} \to P$ there exist $i < j$ such that $f(i) \preceq f(j)$. There are many equivalent classical definitions of well-partial order. In particular a well-partial order is a well founded partial order with no infinite antichains.

- An *order-preserving* map of a partial order $P$ into a partial order $Q$ is a function $f \colon P \to Q$ such that $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for all $x, y \in P$. Notice that an order-preserving map is a one-to-one function.

- An *embedding* of a partial order $P$ into a partial order $Q$ is a function $f \colon P \to Q$ such that $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$ for all $x, y \in P$. Notice that an embedding is a one-to-one function. If $P$ is embeddable into $Q$ we write $P \preceq Q$.

- An *isomorphism* is an onto embedding. If $P$ is isomorphic to $Q$ we write $P \cong Q$.

- Let $\alpha, \beta$ be well-orders. A *strong embedding* of $\alpha$ into $\beta$ is an embedding $f \colon \alpha \to \beta$ such that $\operatorname{ran}(f)$ is an initial interval of $\beta$.

- $P$ is called *scattered* if $\mathbb{Q}$ does not embed into $P$.

- The *inverse* (or *reverse*) of $P$ is $P^* = (P, \succeq)$.

- A *restriction* of $P$ is $S \subseteq P$ equipped with the ordering induced by $P$, namely $x \leq_S y$ if and only if $x \preceq y$ for all $x, y \in S$.

- A *partial extension* (or simply *extension*) of $P$ is a partial order $P' = (P, \preceq')$ such that $x \preceq y$ implies $x \preceq' y$ for all $x, y \in P$.

- If $B$ is a partial order and $\{P_x \colon x \in B\}$ is a sequence of partial orders indexed by $B$ we define the *lexicographic sum* (or simply *sum*) of the $P_x$ along $B$, denoted by $\sum_{x \in B} P_x$, to be the partial order $P$ on the set $\{(x, y) \colon x \in B \wedge y \in P_x\}$ defined by

$$(x, y) \leq_P (x', y') \iff x <_P x' \vee (x = x' \wedge y \leq_{P_x} y').$$

- The *sum* along the $n$-element chain is denoted by $\sum_{i<n} P_i$. Similarly, the *disjoint sum* $\bigoplus_{i<n} P_i$ is the sum along the $n$-element antichain.

**Lemma 1.5.1** ($\mathsf{RCA}_0$). *The following are equivalent:*

(1) $\mathsf{B}\mathbf{\Sigma}^0_2$*;*

(2) *the sum of FAC partial orders along a FAC partial order is FAC.*

*Proof.* (1) $\Rightarrow$ (2). By Theorem 1.3.1, we may assume $\mathsf{RT}^1_{<\infty}$. Let $P = \sum_{x \in B} P_x$ and suppose that $P$ is not FAC. Let $A \subseteq P$ be an infinite antichain. Then the set $\{x \in B \colon (\exists y)(x, y) \in A\}$ is an antichain of $B$. Such a set is $\mathbf{\Sigma}^0_1$ and so might not exist in $\mathsf{RCA}_0$. However, provably in $\mathsf{RCA}_0$, any infinite $\mathbf{\Sigma}^0_1$ set contains an infinite $\mathbf{\Delta}^0_1$ set, and hence we may assume that such a set is finite, since otherwise we could define an infinite antichain of $B$. Let $n \in \mathbb{N}$ be such that $(x, y) \in A$ implies $x < n$. Fix a one-to-one enumeration $g \colon \mathbb{N} \to \mathbb{N}$ of $A$. Define $f \colon \mathbb{N} \to n$ by letting $f(i) = x$ where $g(i) = (x, y) \in A$. By $\mathsf{RT}^1_{<\infty}$, there exists $x < n$ such that $\{(x, y) \in A \colon y \in \mathbb{N}\}$ is infinite. It follows that $\{y \in \mathbb{N} \colon (x, y) \in A\}$ is an infinite antichain of $P_x$ and $P_x$ is not FAC.

(2) $\Rightarrow$ (1). By Theorem 1.3.1 again, we prove $\mathsf{RT}^1_{<\infty}$. Let $f \colon \mathbb{N} \to n$ be a function and define the partial order $P = \bigoplus_{i<n} P_i$, where each $P_i = \{x \in \mathbb{N} \colon f(x) = i\}$ is viewed as an antichain. Suppose that $P_i$ is finite for all $i < n$. Therefore, $P$ is the sum of FAC partial orders along a FAC partial order and by (2) is FAC. Then $P$ is finite, which is a contradiction. $\square$

**Lemma 1.5.2** ($\mathsf{RCA}_0$). *The following hold:*

(1) *every sum of scattered partial orders along a scattered partial order is scattered;*

(2) *the reverse of a scattered partial order is scattered;*

(3) *every well-order is scattered;*

(4) *every well-partial order is scattered.*

*Proof.* (1) Let $P = \sum_{x \in B} P_x$ and suppose that $P$ is not scattered. Fix an embedding $f \colon \mathbb{Q} \to P$. First suppose that for some $i <_{\mathbb{Q}} j$ and $x \in P$ we have $f(i) = (x, y)$ and $f(j) = (x, z)$. Then, the composition of $f$ with the projection on the second coordinate is an embedding of the rational interval $(i, j)_{\mathbb{Q}}$ into $P_x$. Since $\mathbb{Q}$ embeds into its open intervals, $P_x$ is not scattered. Otherwise, composing $f$ with the projection on the first coordinate, we obtain an embedding of $\mathbb{Q}$ into $B$, and $P$ is not scattered.

(2) Let $P$ be a partial order and suppose that $\mathbb{Q}$ embeds into $P^*$ via $f$. Since $\mathbb{Q}$ embeds into any dense linear order, let $g$ be an embedding of $\mathbb{Q}$ into $\mathbb{Q}^*$. Therefore $f \circ g$ is an embedding of $\mathbb{Q}$ into $P$ and $P$ is not scattered.

(3) Suppose $P$ is not scattered and let $f \colon \mathbb{Q} \to P$ be an embedding. Composing $f$ with a descending sequence of $\mathbb{Q}$, we obtain a descending sequence of $P$.

(4) follows from (3). $\square$

### 1.5.2 Trees

We use $\sigma, \tau, \eta, \ldots$ to denote finite sequences of natural numbers, that is elements of $\mathbb{N}^{<\mathbb{N}}$. The set of finite binary sequences is denoted by $2^{<\mathbb{N}}$.

- Let $|\sigma|$ be the *length* of $\sigma$ and list it as $\langle \sigma(0), \ldots, \sigma(|\sigma| - 1) \rangle$. In particular $\langle\rangle$ is the unique sequence of length $0$.
- We write $\sigma \subseteq \tau$ to mean that $\sigma$ is an *initial segment* of $\tau$, while $\sigma^\frown\tau$ denotes the *concatenation* of $\sigma$ and $\tau$.
- By $\sigma \restriction k$ we mean the initial segment of $\sigma$ of length $k$ and similarly, when $f$ is a function, $f \restriction k$ is the finite sequence $\langle f(0), \ldots, f(k-1) \rangle$.
- Let $\sigma \cap \tau$ be the longest common initial segment of $\sigma$ and $\tau$, that is $\sigma \restriction k$, where $k$ is unique such that $(\forall i < k) \sigma(i) = \tau(i)$ and $\sigma(k) \neq \tau(k)$.
- A *tree* $T$ is a set of finite sequences such that $\tau \in T$ and $\sigma \subseteq \tau$ imply $\sigma \in T$. A *path* in $T$ is a function $f$ such that for all $n$ the finite sequence $f \restriction n$ belongs to $T$.

- We write $[T]$ to denote the collection of all paths in $T$: $[T]$ does not formally exists in second order arithmetic but $f \in [T]$ is a convenient shorthand.
- A tree $T$ is *perfect* if for all $\sigma \in T$ there exist $\tau_0, \tau_1 \in T$ such that $\sigma \subseteq \tau_0, \tau_1$ and neither $\tau_0 \subseteq \tau_1$ nor $\tau_1 \subseteq \tau_0$ hold.
- A tree $T$ has *countably many paths* if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ (coded by a single set) such that for every $f \in [T]$ there exists $n \in \mathbb{N}$ such that $f = f_n$. If $T$ does not have countably many paths then we say that it has *uncountably many paths*.
- The *Kleene-Brouwer order* on finite sequences is the linear order defined by $\sigma \leq_{\mathrm{KB}} \tau$ if either $\tau \subseteq \sigma$ or $\sigma(i) < \tau(i)$ for the least $i$ such that $\sigma(i) \neq \tau(i)$.
- If $T$ is a tree, let $\mathrm{KB}(T)$ be the restriction of $\leq_{\mathrm{KB}}$ to $T$.

By [Sim09, Theorem V.5.5] $\mathsf{ATR}_0$ is equivalent to the perfect tree theorem, stating that every tree with uncountably many paths contains a perfect subtree. The following straightforward diagonal argument shows in $\mathsf{RCA}_0$ that a nonempty perfect tree has uncountably many paths.

**Lemma 1.5.3** ($\mathsf{RCA}_0$)**.** *A non-empty perfect tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has uncountably many paths.*

*Proof.* Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a non-empty perfect tree and $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions. We aim to prove that there exists a path $f$ of $T$ such that $(\forall n)(f \neq f_n)$. By recursion, we define a sequence of elements $\sigma_n \in T$ such that $\sigma_n \subseteq \sigma_{n+1}$ and for all $n \in \mathbb{N}$

$$(\exists i < |\sigma_{n+1}|)(\sigma_{n+1}(i) \neq f_n(i)).$$

Let $\sigma_0 = \langle\rangle$. To define $\sigma_{n+1}$, search for the $\omega$-least triple $\langle \tau, p, q\rangle \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$ such that $\sigma_n \subseteq \tau$, $\tau^\frown\langle p\rangle, \tau^\frown\langle q\rangle \in T$ and $f_n(|\tau|) \neq p$. Then, let $\sigma_{n+1} = \tau^\frown\langle p\rangle$.

Since $T$ is perfect, we always find such triples. Now, the function $f = \bigcup_{n\in\mathbb{N}} \sigma_n$ is $\mathbf{\Delta}^0_1$ definable and is as desired. $\square$

The main feature of $\leq_{\mathrm{KB}}$ is that, provably in $\mathsf{ACA}_0$, its restriction to a tree $T$ is a well-order if and only if $T$ has no paths ([Sim09, Lemma V.1.3]).

**Theorem 1.5.4** ([Hir94])**.** *Over* $\mathsf{RCA}_0$, $\mathsf{ACA}_0$ *is equivalent to the statement "a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has no paths if and only if $\mathrm{KB}(T)$ is well-ordered."*

We notice that the above theorem is not explicitly stated and proved in [Hir94], although the idea of the reversal is already contained in the proof of [Hir94, Theorem 2.6]. To be precise, Hirst shows that $\mathsf{ACA}_0$ is equivalent to the statement

$$(\forall f)(\mathrm{KB}(T(f)) \text{ is not well-founded}),$$

where $T(f)$ is a finitely branching tree which is $\boldsymbol{\Delta}^0_1$ definable (in $f$). The key point is that, provably in $\mathsf{RCA}_0$, any descending sequence through $\mathrm{KB}(T(f))$ computes a path in $T(f)$ and any path of $T(f)$ computes $\mathrm{ran}(f)$ (see also [Sim09, Theorem III.7.2]). We thus provide a proof of the reversal.

*Proof.* By Theorem 1.3.3, we show that the range of a given one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$ exists. Define $T = T(f) \subseteq \mathbb{N}^{<\mathbb{N}}$ by $\tau \in T$ if and only if for all $m < |\tau|$:

- $\tau(m) > 0 \implies f(\tau(m) - 1) = m$ and
- $\tau(m) = 0 \implies (\forall n < |\tau|) f(n) \neq m$.

It is clear that $T$ has at most one path and that if $h$ is a path in $T$, then $m \in \mathrm{ran}(f)$ if and only if $h(m) > 0$.

We modify $T$ as follows. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let $\mathrm{evn}(\sigma) = \langle \sigma_0, \sigma_2, \sigma_4, \ldots \rangle$ and $\mathrm{odd}(\sigma) = \langle \sigma_1, \sigma_3, \sigma_5 \ldots \rangle$. Thus, let $T^* = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \colon \mathrm{evn}(\sigma) \in 2^{<\mathbb{N}} \wedge \mathrm{odd}(\sigma) \in T\}$.

Notice that a path in $T^*$ yields a path in $T$. In fact, if $g \in [T^*]$ then $\mathrm{odd}(g) \in [T]$, where $\mathrm{odd}(g)(n) = g(2n+1)$ for all $n$.

We aim to show that $\mathrm{KB}(T^*)$ is not well-ordered.

For $\tau \in T$, let $\tau^*$ be the unique $\sigma \in T^*$ such that $|\sigma|$ is even, $\mathrm{odd}(\sigma) = \tau$ and $\mathrm{evn}(\sigma)(m) = 0 \Leftrightarrow \tau(m) > 0$ for all $m < |\tau|$. For all $k \in \mathbb{N}$, we define $\tau_k \in T$ of length $k$ by letting, for all $m < k$, $\tau_k(m) = 0$ if $f(n) \neq m$ for all $n < |\sigma|$, $\tau_k(m) = n + 1$ if $n < k$ and $f(n) = m$ (since $f$ is one-to-one, $n$ is unique). It is not difficult to see that $(\tau^*_k)$ is a descending sequence through $\mathrm{KB}(T^*)$. $\square$

## Note on finite sets

A set $F$ is *finite* if $(\exists k)(\forall n)(n \in F \implies n < k)$. Within $\mathsf{RCA}_0$, one can prove that any finite set $F$ has a unique *code*, a natural number $n$ such that $(\forall i)(i \in X \Leftrightarrow \theta(i, n))$, where $\theta(i, n)$ is a $\boldsymbol{\Sigma}^0_0$ formula (see [Sim09, Theorem II.2.5]). Besides, checking that $n \in \mathbb{N}$ is a code is $\boldsymbol{\Sigma}^0_0$. From now on, we sometimes identify a finite set $F$ with its code so that a formula containing only set variables for finite sets is actually arithmetical.

# 2

# Antichains and initial intervals[1]

## 2.1 Introduction

The following theorem can be found in Fraïssé's monograph [Fra00, § 4.7.2], where it is attributed to Bonnet [Bon75].

**Theorem 2.1.1.** *A partial order is FAC if and only if every initial interval is a finite union of ideals.*

In [PS06] Theorem 2.1.1 is attributed to Erdös and Tarski because its 'hard' (left to right) direction can be deduced from the following result, which is part of [ET43, Theorem 1]:

**Theorem 2.1.2.** *If a partial order $P$ has no infinite strong antichains then there are no arbitrarily large finite strong antichains in $P$.*

(One should notice that Erdös and Tarski work with what we would call filters and final intervals.)

An intermediate step between Theorems 2.1.2 and 2.1.1 is the following characterization of partial orders with no infinite strong antichains:

**Theorem 2.1.3.** *A partial order $P$ has no infinite strong antichains if and only if $P$ is a finite union of ideals.*

Since Theorems 2.1.1 and 2.1.3 are equivalences, we study separately the two implications, which turn out to have different axiomatic strengths.

In section 2.3, we prove, over $\mathsf{RCA}_0$, the equivalence of $\mathsf{ACA}_0$ with each of the following statements:

[1]The content of this chapter appears in [FM14]

(1) in a countable partial order with no infinite antichains every initial interval is a finite union of ideals;

(2) in a countable partial order with no infinite strong antichains there is a bound on the size of the strong antichains;

(3) every countable partial order with no infinite strong antichains is a finite union of ideals.

In the last two sections we deal with the "easy" (right to left) direction of Theorem 2.1.1. In section 2.4, we show that the statement is provable in $\mathsf{WKL}_0$ (the obvious proof goes through $\mathsf{ACA}_0$). Finally, in section 2.5, we show that the statement fails in the $\omega$-model of computable sets and hence cannot be proved in $\mathsf{RCA}_0$. Our results do not settle the reverse mathematics strength of the statement, leaving open the possibility that it lies between $\mathsf{RCA}_0$ and $\mathsf{WKL}_0$. On the other hand, $\mathsf{RCA}_0$ easily suffices to show that every countable partial order which is a finite union of ideals has no infinite strong antichains (Lemma 2.4.1).

## 2.2 Preliminaries

We recall the following definitions from subsection 1.5.1.

**Definition 2.2.1.** Let $P$ be a partial order. A subset $A \subseteq P$ is an *antichain* if all its elements are pairwise incomparable, i.e.

$$(\forall x, y \in A)(x \neq y \implies x \npreceq y \land y \npreceq x).$$

A subset $S \subseteq P$ is a *strong antichain* in $P$ if its elements are pairwise incompatible in $P$, i.e.

$$(\forall x, y \in S)(\forall z \in P)(x, y \preceq z \implies x = y).$$

A set $I \subseteq P$ is an *initial interval* of $P$ if

$$(\forall x, y \in P)(x \preceq y \land y \in I \implies x \in I).$$

An initial interval $A$ of $P$ is an *ideal* if every two elements of $A$ are compatible in $A$, i.e.

$$(\forall x, y \in A)(\exists z \in A)(x \preceq z \land y \preceq z).$$

### 2.2.1 Essential union of sets

In this section we prove a technical result that is useful in the remainder of the chapter. This result deals with finite union of sets and will be applied to finite unions of ideals.

**Definition 2.2.2** ($\mathsf{RCA}_0$)**.** Let $I \subseteq \mathbb{N}$. A family of sets $\{A_i : i \in I\}$ is *essential* if

$$(\forall i \in I)(A_i \nsubseteq \bigcup_{j \in I, j \neq i} A_j).$$

The union of such a family is called an *essential union*.

Not every family of sets can be made essential without loosing elements from the union. The simplest example is a sequence $\{A_n : n \in \mathbb{N}\}$ of sets such that $A_n \subset A_{n+1}$ for every $n$. However the following shows that, provably in $\mathsf{RCA}_0$, every finite family of sets can be made essential.

**Lemma 2.2.3** ($\mathsf{RCA}_0$)**.** *For every family of sets $\{A_i : i \in F\}$ with $F$ finite there exists $I \subseteq F$ such that $\{A_i : i \in I\}$ is essential and*

$$\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i.$$

*Proof.* Let $n$ be $\omega$-least such that there exists (a code of) a finite set $I$ such that $I \subseteq F$, $|I| = n$ and $\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i$. One can check that such property is $\mathbf{\Pi}^0_1$ (see also subsection 1.5.2). By $\mathbf{\Sigma}^0_1$-induction, since the code of $F$ satisfies such property, $n$ exists in $\mathsf{RCA}_0$. Let $I \subseteq F$ be a witness of $n$. Then $\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i)$. Moreover, by the minimality of $n$, it is easy to see that $\{A_i : i \in I\}$ is essential. $\square$

### 2.2.2 $\mathsf{WKL}_0$ and initial interval separation

The following provides an equivalence with $\mathsf{WKL}_0$, inspired by the usual $\mathbf{\Sigma}^0_1$ separation.

**Lemma 2.2.4.** *Over $\mathsf{RCA}_0$, the following are equivalent:*

(1) $\mathsf{WKL}_0$*;*

(2) $\mathbf{\Sigma}^0_1$ *initial interval separation. Let $P$ be a partial order and $\varphi(x)$, $\psi(x)$ be $\mathbf{\Sigma}^0_1$ formulas with one distinguished free number variable.*

*If* $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \not\preceq x)$*, then there exists an initial interval* $I$ *of* $P$ *such that*

$$(\forall x \in P)((\varphi(x) \implies x \in I) \wedge (\psi(x) \implies x \notin I)).$$

(3) *Initial interval separation. Let* $P$ *be a partial order and suppose* $A, B \subseteq P$ *are such that* $(\forall x \in A)(\forall y \in B) y \not\preceq x$*. Then there exists an initial interval* $I$ *of* $P$ *such that* $A \subseteq I$ *and* $B \cap I = \emptyset$*.*

*Proof.* We first assume $\mathsf{WKL}_0$ and prove (2). Fix the partial order $P$ and let $\varphi(x) \equiv (\exists m)\varphi_0(x, m)$ and $\psi(n) \equiv (\exists m)\psi_0(x, m)$ be $\mathbf{\Sigma}^0_1$ formulas with $\varphi_0$ and $\psi_0$ $\mathbf{\Sigma}^0_0$. Assume $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \not\preceq x)$.

Form the binary tree $T \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in T$ if and only if $\sigma \in T(P)$ and for all $x, y < |\sigma|$:

(i) $(\exists m < |\sigma|)\, \varphi_0(x, m) \implies \sigma(x) = 1$, and

(ii) $(\exists m < |\sigma|)\, \psi_0(x, m) \implies \sigma(x) = 0$.

To see that $T$ is infinite, we show that for every $k \in \mathbb{N}$ there exists $\sigma \in T$ with $|\sigma| = k$. Given $k$ let

$$\sigma(x) = 1 \iff x \in P \wedge (\exists y, m < k)(\varphi_0(y, m) \wedge x \preceq y)$$

for all $x < k$. It is easy to verify that $\sigma \in T$. By weak König's lemma, $T$ has a path $f$. By $\mathbf{\Sigma}^0_0$ comprehension, let $I = \{x \colon f(x) = 1\}$. It is straightforward to see that $I$ is as desired.

(3) is the special case of (2) obtained by considering the $\mathbf{\Sigma}^0_0$, and hence $\mathbf{\Sigma}^0_1$, formulas $x \in A$ and $x \in B$.

It remains to prove (3) $\Rightarrow$ (1). It suffices to derive in $\mathsf{RCA}_0$ from (3) the existence of a set separating the disjoint ranges of two one-to-one functions (see Theorem 1.3.2). Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $(\forall n, m \in \mathbb{N}) f(n) \neq g(m)$. Define a partial order on $P = \{a_n, b_n, c_n \colon n \in \mathbb{N}\}$ by letting $c_n \preceq a_m$ if and only if $f(m) = n$, $b_m \preceq c_n$ if and only if $g(m) = n$, and adding no other comparabilities. Let $A = \{a_n \colon n \in \mathbb{N}\}$ and $B = \{b_n \colon n \in \mathbb{N}\}$, so that $(\forall x \in A)(\forall y \in B) y \not\preceq x$. By (3) there exists an initial interval $I$ of $P$ such that $A \subseteq I$ and $B \cap I = \emptyset$. It is easy to check that $\{n \colon c_n \in I\}$ separates the range of $f$ from the range of $g$. □

### 2.2.3 Decomposition into ideals

In this section we prove a technical result about countable partial orders. We first notice that every partial order is (provably in $\mathsf{RCA}_0$) a union of ideals: consider, for instance, the principle ideals. Here we show that for every countable partial order there is a countable collection of ideals such that the partial order is a finite union of ideals exactly when such collection is finite. Moreover, there is an effective procedure (arithmetical in the partial order) to produce such collection of ideals.

**Lemma 2.2.5** ($\mathsf{ACA}_0$)**.** *Let $P$ be a partial order. Then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of elements of $P^{<\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and for all $i < |\sigma_n|$:*

(1) $|\sigma_n| \leq |\sigma_{n+1}|$*;*

(2) $\sigma_n(i) \preceq \sigma_{n+1}(i)$*;*

(3) $\{\sigma_n(i) \colon i < |\sigma_n|\}$ *is a strong antichain;*

(4) $\{\sigma_n(i) \colon n \in \mathbb{N}, i < |\sigma_n|\}$ *is cofinal in $P$.*

*Proof.* Let $P = \{x_n \colon n \in \mathbb{N}\}$ be an infinite partial order. By arithmetical recursion we define $\sigma_n \in P^{<\mathbb{N}}$ for all $n$. We let $\sigma_0 = \langle\rangle$. Assume that we have already defined $\sigma_n$ and consider $x_n$. If

$$(\exists i < |\sigma_n|)(x_n \text{ is compatible with } \sigma_n(i)),$$

let $(i, z)$ be the least natural number such that $i < |\sigma_n|$, $z \in P$ and both $x_n$ and $\sigma_n(i)$ are $\preceq z$. Let $|\sigma_{n+1}| = |\sigma_n|$, $\sigma_{n+1}(i) = z$, and $\sigma_{n+1}(j) = \sigma_n(j)$ for $j \neq i$. Otherwise, let $\sigma_{n+1} = \sigma_n{}^\frown\langle x_n\rangle$.

It is clear that $(1)$ and $(2)$ hold. By arithmetical induction it is straightforward to verify condition $(3)$. By construction, for all $n \in \mathbb{N}$ there is $i < |\sigma_{n+1}|$ such that $x_n \preceq \sigma_{n+1}(i)$, and then $(4)$ also holds. $\square$

**Corollary 2.2.6** ($\mathsf{ACA}_0$)**.** *Let $P$ be a partial order. Then there exists a family $\{A_i \colon i \in I\}$ of ideals of $P$ such that:*

$(a)$ $P = \bigcup_{i\in I} A_i$*;*

$(b)$ *$I$ is finite if and only if there is a finite bound on the size of strong antichains in $P$.*

*In particular, $P$ is a finite union of ideals if and only if $I$ is finite.*

*Proof.* Let $P$ be a partial order and $(\sigma_n)_{n\in\mathbb{N}}$ be as in Lemma 2.2.5. We then define in $\mathsf{ACA}_0$ a set $\{A_i : i \in \mathbb{N}\}$ of subsets of $P$ by letting:

$$A_i = \{x \in P : (\exists n)(i < |\sigma_n| \wedge x \preceq \sigma_n(i)\}$$

for all $i \in \mathbb{N}$. By conditions (1) and (2) of Lemma 2.2.5, every $A_i$ is an ideal of $P$. If we let $I = \{i \in \mathbb{N} : (\exists n)(i < |\sigma_n|)\}$, it follows by (4) that $P = \bigcup_{i\in I} A_i$. Hence, (a) holds.

We now show (b). If there are arbitrarily large finite strong antichains, then $P$ cannot be the union of finitely many ideals, for otherwise we would find two incompatible elements in the same ideal, which is clearly a contradiction, and so $I$ must be infinite. On the other hand, if $I$ is infinite, then by (3) the sequence $(\sigma_n)_{n\in\mathbb{N}}$ provides arbitrarily large finite strong antichains. □

## 2.3 Equivalences with $\mathsf{ACA}_0$

We consider the following equivalence, which includes Theorems 2.1.2 and 2.1.3.

**Theorem 2.3.1.** *Let $P$ be a countable partial order. Then the following are equivalent:*

(1) *$P$ is a finite union of ideals;*

(2) *there is a finite bound on the size of the strong antichains in $P$;*

(3) *there is no infinite strong antichain in $P$.*

We notice that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are easily provable in $\mathsf{RCA}_0$. We show that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are provable in $\mathsf{ACA}_0$. Note that (3) $\Rightarrow$ (2) is false if we consider antichains in place of strong antichains.

We start with implication (2) $\Rightarrow$ (1).

**Lemma 2.3.2** ($\mathsf{ACA}_0$)**.** *Let $P$ be a partial order with no arbitrarily large finite strong antichains. Then $P$ is a finite union of ideals.*

*Proof.* Let $\ell \in \mathbb{N}$ be the maximum size of a strong antichain in $P$ and let $S$ be a strong antichain of size $\ell$. For every $z \in S$ define by arithmetical comprehension

$$A_z = \{x \in P : x \text{ and } z \text{ are compatible}\}.$$

Since $S$ is maximal with respect to inclusion it is immediate that $P = \bigcup_{z \in S} A_z$ and it suffices to show that each $A_z$ is an ideal.

Fix $z \in S$ and $x, y \in A_z$. Let $x_0, y_0$ be such that $x \preceq x_0$, $y \preceq y_0$, and $z \preceq x_0, y_0$. It suffices to show that $x_0$ and $y_0$ are compatible in $A_z$. If this is not the case, $x_0$ and $y_0$ are incompatible also in $P$ (because $P(\succeq x_0) \subseteq P(\succeq z) \subseteq A_z$). Moreover for each $w \in S \setminus \{z\}$ each of $x_0$ and $y_0$ is incompatible with $w$ in $P$ because $z$ and $w$ are incompatible in $P$. Thus $(S \setminus \{z\}) \cup \{x_0, y_0\}$ is a strong antichain of size $\ell + 1$, a contradiction. $\square$

To obtain $(3) \Rightarrow (2)$ of Theorem 2.3.1 we are going to use the existence of maximal (with respect to inclusion) strong antichains. We first show that this statement is equivalent to $\mathsf{ACA}_0$.

**Lemma 2.3.3.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

$(1)$ $\mathsf{ACA}_0$*;*

$(2)$ *every strong antichain in a partial order extends to a maximal strong antichain;*

$(3)$ *every partial order contains a maximal strong antichain.*

*Proof.* We show $(1) \Rightarrow (2)$. Let $P$ be a partial order and $S \subseteq P$ a strong antichain. By primitive recursion, we define a maximal strong antichain $T \supseteq S$ in $P$. Suppose we have defined $T_y = T \cap \{x \in P \colon x < y\}$. Then $y \in T$ if and only if $S \cup T_y \cup \{y\}$ is a strong antichain.

Implication $(2) \Rightarrow (3)$ is trivial. To show $(3) \Rightarrow (1)$, we argue in $\mathsf{RCA}_0$ and derive from (3) the existence of the range of any one-to-one function. Given $f \colon \mathbb{N} \to \mathbb{N}$ one-to-one consider $P = \{a_n, b_n, c_n \colon n \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ let $a_n \preceq c_m$ if and only if $b_n \preceq c_m$ if and only if $f(m) = n$, and add no other comparabilities. By (3), let $S \subseteq P$ be a maximal strong antichain. Then, $n$ belongs to the range of $f$ if and only if $a_n \notin S \vee b_n \notin S$, and the range of $f$ exists by $\mathbf{\Sigma}^0_0$ comprehension. $\square$

The following is implication $(3) \Rightarrow (2)$ of Theorem 2.3.1, i.e. our formalization of the left to right direction of Theorem 2.1.2.

**Lemma 2.3.4** ($\mathsf{ACA}_0$)**.** *Let $P$ be a partial order with no infinite strong antichains. Then there are no arbitrarily large finite strong antichains in $P$.*

*Proof.* Suppose for a contradiction that $P$ has arbitrarily large finite strong antichains but no infinite strong antichains (the existence of such a pair is proved below). We define by recursion a sequence of elements $(x_n, y_n) \in P^2$.

Let $(x_0, y_0)$ be a pair such that $x_0$ and $y_0$ are incompatible in $P$ and $P(\succeq x_0)$ contains arbitrarily large finite strong antichains. Suppose we have defined $x_n$ and $y_n$. Using arithmetical comprehension, search for a pair $(x_{n+1}, y_{n+1})$ such that $x_n \preceq x_{n+1}, y_{n+1}$, $x_{n+1}$ and $y_{n+1}$ are incompatible in $P$, and $P(\succeq x_{n+1})$ contains arbitrarily large finite strong antichains.

To show that the recursion never stops assume that $U \subseteq P$ is a final interval with arbitrarily large finite strong antichains ($U = P$ at stage $0$, $U = P(\succeq x_n)$ at stage $n+1$). By Lemma 2.3.3 there exists a maximal strong antichain $S \subseteq U$ with at least two elements. By hypothesis, $S$ is finite and we apply the following claim:

**Claim.** There exists $x \in S$ such that $P(\succeq x)$ contains arbitrarily large finite strong antichains.

*Proof of claim.* Let $n = |S|$. We first show that for every $k \geq 1$ there exists $u \in S$ such that $P(\succeq u)$ contains a strong antichain of size $k$.

Given $k \geq 1$, let $T$ be a strong antichain of size $n \cdot k$. Since $S$ is maximal, every element $y \in T$ is compatible with some element of $S$. For any $y \in T$ let $(u(y), v(y))$ be the least pair such that $u(y) \in S$ and $u(y), y \preceq v(y)$. Then $\{v(y) \colon y \in T\}$ is again a strong antichain of size $n \cdot k$. As $y \mapsto u(y)$ defines a function from $T$ to $S$, it easily follows that for some $u \in S$ the upper cone $P(\succeq u)$ contains at least $k$ elements of the form $v(y)$ with $y \in T$.

Now, for all $k \geq 1$, let $u_k \in S$ be such that $P(\succeq u_k)$ contains a strong antichain of size $k$. Since $S$ is finite, by the infinite pigeonhole principle (which is provable in $\mathsf{ACA}_0$), there exists $x \in S$ such that $x = u_k$ for infinitely many $k$. The upper cone $P(\succeq x)$ thus contains arbitrarily large finite strong antichains. □

In particular, $x_n \preceq y_m$ for all $n < m$ and $x_n$ and $y_n$ are incompatible in $P$. It follows that $y_n$ is incompatible with $y_m$ for all $n < m$. Then $\{y_n \colon n \in \mathbb{N}\}$ is an infinite strong antichain, for the desired contradiction. □

The following Theorem shows that our use of $\mathsf{ACA}_0$ in several of the preceding Lemmas is necessary and establish the reverse mathematics results about Theorem 2.1.2 and the left to right directions of Theorems 2.1.1 and 2.1.3 (these are respectively conditions (3), (5), and (4) in the statement of the Theorem). We also show that apparently weaker statements, such as the restriction of Theorems 2.1.1 and 2.1.3 to well-partial orders, require $\mathsf{ACA}_0$.

**Theorem 2.3.5.** *Over $\mathsf{RCA}_0$, the following are pairwise equivalent:*

(1) $\mathsf{ACA}_0$*;*

(2) *every partial order with no arbitrarily large finite strong antichains is a finite union of ideals;*

(3) *every partial order with no infinite strong antichains does not contain arbitrarily large finite strong antichains;*

(4) *every partial order with no infinite strong antichains is a finite union of ideals;*

(5) *if a partial order is FAC then every initial interval is a finite union of ideals;*

(6) *every well-partial order is a finite union of ideals.*

*Proof.* $(1) \Rightarrow (2)$ is Lemma 2.3.2 and $(1) \Rightarrow (3)$ is Lemma 2.3.4. The combination of Lemma 2.3.4 and Lemma 2.3.2 shows $(1) \Rightarrow (4)$. Since a strong antichain in a subset of a partial order is an antichain, $(4) \Rightarrow (5)$ holds. For $(5) \Rightarrow (6)$, recall that, provably in $\mathsf{RCA}_0$, a well-partial order has no infinite antichains.

It remains to show that each of (2), (3) and (6) implies $\mathsf{ACA}_0$. Reasoning in $\mathsf{RCA}_0$ fix a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$. In each case we build a suitable partial order $P$ which encodes the range of $f$.

We start with $(2) \Rightarrow (1)$. Let $P = \{a_n, b_n \colon n \in \mathbb{N}\} \cup \{c\}$. We define a partial order on $P$ by letting:

- $a_n \preceq c$ for all $n$;
- $b_n \preceq b_m$ for $n \leq m$;
- $a_n \preceq b_m$ if and only if $(\exists i < m) f(i) = n$;

and adding no other comparabilities. It is easy to verify that every strong antichain in $P$ has at most 2 elements. By (2) $P$ is a finite union of ideals $A_0, \dots, A_k$. By Lemma 2.2.3, we may assume that this union is essential. Let us assume $b_0 \in A_0$.

By $\mathbf{\Sigma}^0_1$-induction (actually $\mathbf{\Sigma}^0_0$) we prove that $(\forall m)(b_m \in A_0)$. The base step is obviously true. Suppose $b_m \in A_0$ and $b_{m+1} \notin A_0$. Then $A_0 = \{x \in P \colon x \preceq b_m\}$ (because every element $\succ b_m$ is $\succeq b_{m+1}$). Suppose $b_{m+1} \in A_1$. Then $A_0 \subseteq A_1$ and the decomposition is not essential, a contradiction. Therefore, $A_0$ contains all the $b_m$'s. Now, it is straightforward to see that $(\exists m) f(m) = n$ if and only if $a_n \in A_0$, so that the range of $f$ can be defined by $\mathbf{\Delta}^0_0$ comprehension.

To prove $(3) \Rightarrow (1)$ we exploit the notion of false and true stage. Recall that $n \in \mathbb{N}$ is said to be a *false stage for $f$* (or simply *false*) if $f(k) < f(n)$ for some $k > n$ and *true* otherwise. We may assume to have infinitely many false stages, since otherwise the range of $f$ exists by $\mathbf{\Delta}^0_1$ comprehension. On the other hand, there are always infinitely many true stages (i.e. for every $m$ there exists $n > m$ which is true), because otherwise we can build an infinite descending sequence of natural numbers.

Let $P = \{a_n, b_n \colon n \in \mathbb{N}\}$ and define

- $b_n \preceq b_m$ for all $n < m$;
- $a_n \preceq b_m$ if and only if $f(k) < f(n)$ for some $k$ with $n < k \leq m$ (i.e. if at stage $m$ we know that $n$ is false);

and there are no other comparabilities.

Notice that the $b_n$'s and the $a_n$'s with $n$ false are pairwise compatible in $P$. Therefore every infinite strong antichain in $P$ consists of infinitely many $a_n$'s with $n$ true and at most one $b_n$ or $a_n$ with $n$ false. Possibly removing that single element we have an infinite set of true stages. From this in $\mathsf{RCA}_0$ we can obtain a strictly increasing enumeration of true stages $i \mapsto n_i$. Since $(\exists n) f(n) = m$ if and only if $(\exists n \leq n_m) f(n) = m$, the range of $f$ exists by $\mathbf{\Delta}^0_1$ comprehension. Thus the existence of an infinite strong antichain in $P$ implies the existence of the range of $f$ in $\mathsf{RCA}_0$.

To apply $(3)$ and conclude the proof we need to show that $P$ contains arbitrarily large finite strong antichains. To do this apparently we need $\mathbf{\Sigma}^0_2$-induction (which is not available in $\mathsf{RCA}_0$) to show that for all $k$ there exists $k$ distinct true stages.

To remedy this problem (with the same trick used for this purpose in [MS11, Lemma 4.2]) we replace each $a_n$ with $n+1$ distinct elements. Thus we set $P' = \{a^i_n, b_n \colon n \in \mathbb{N}, i \leq n\}$ and substitute (ii) with $a^i_n \leq_{P'} b_m$ if and only if $f(k) < f(n)$ for some $k$ with $n < k \leq m$. Then also the existence of an infinite strong antichain in $P'$ suffices to define the range of $f$ in $\mathsf{RCA}_0$. However the existence of arbitrarily large finite strong antichains in $P'$ of the form $\{a^i_n \colon i \leq n\}$ follows immediately from the existence of infinitely many true stages.

We now show $(6) \Rightarrow (1)$. We again use false and true stages and as before we assume to have infinitely many false stages. The idea for $P$ is to combine a linear order $P_0 = \{a_n : n \in \ N\}$ of order type $\omega + \omega^*$ with a linear order $P_1 = \{b_n : n \in \mathbb{N}\}$ of order type $\omega$. The false and true stages give rise respectively to the $\omega$ and $\omega^*$ part of $P_0$, and every false stage is below some element of $P_1$. We proceed as follows.

Let $P = \{a_n, b_n : n \in \mathbb{N}\}$. For $n \leq m$, set

(i) $a_n \preceq a_m$ if $f(k) < f(n)$ for some $n < k \leq m$ (i.e. if at stage $m$ we know that $n$ is false);

(ii) $a_m \preceq a_n$ if $f(k) > f(n)$ for all $n < k \leq m$ (i.e. if at stage $m$ we believe $n$ to be true).

When condition (i) holds, we also put $a_n \preceq b_m$. Then we linearly order the $b_m$'s by putting $b_i \preceq b_j$ if and only if $i \leq j$. There are no other comparabilities.

It is not difficult to verify that $P$ is a partial order with no infinite antichains. Note that if $n$ is false and $m > n$ is such that $f(m) < f(n)$, then $\{i : a_i \preceq a_n\} \subseteq \{i : i < m\}$ is finite, while if $n$ is true, then $\{i : a_n \preceq a_i\} \subseteq \{i : i \leq n\}$ is finite. This explains our assertion that $P_0$ has order type $\omega + \omega^*$.

First assume that $P$ is not a well-partial order. By definition, there exists $g : \mathbb{N} \to P$ such that $i < j$ implies $g(i) \not\preceq g(j)$. As for every false $n$ there are only finitely many $x \in P$ such that $a_n \not\preceq x$, we must have $g(i) \neq a_n$ for all $i$ and for all false $n$. We may assume that $g(i) \neq b_n$ for all $i, n$, since there are finitely many $b_m$ such that $b_n \not\preceq b_m$. We thus have $g(i) = a_{n_i}$ with $n_i$ true for all $i$. Since $a_m \succ a_n$ and $n < m$ imply $n$ false, the map $i \mapsto n_i$ is a strictly increasing enumeration of true stages. As before, the range of $f$ exists by $\mathbf{\Delta}^0_1$ comprehension.

We now assume that $P$ is a well-partial order. Apply $(6)$, so that $P = \bigcup\{A_i : i < k\}$ is a finite union of ideals. By Lemma 2.2.3 we may assume that the union is essential so that there exists an ideal, say $A_0$, that contains all the $b_m$'s.

We claim that $n$ is false if and only if $a_n \in A_0$. To see this, let $n$ be false. Thus $a_n \preceq b_m$ for some $m$, and hence $a_n \in A_0$. Conversely, if $a_n \in A_0$ then it is compatible with, for instance, $b_0$, and yet again it is $\preceq b_m$ for some $m$. Hence, the set of true stages is $\{n : a_n \notin A_0\}$, and the conclusion follows as before. $\square$

## 2.4 Proofs in $\mathsf{WKL}_0$

We start with a simple observation about the right to left direction of Theorem 2.1.3.

**Lemma 2.4.1** ($\mathsf{RCA}_0$). *Every partial order which is a finite union of ideals has no infinite strong antichains.*

*Proof.* Since an ideal does not contain incompatible elements if the partial order is the union of $k$ ideals we have even a finite bound on the size of strong antichains. $\square$

We now look at the right to left direction of Theorem 2.1.1, which states that every partial order with an infinite antichain contains an initial interval that cannot be written as a finite union of ideals. The proof can be carried out very easily in $\mathsf{ACA}_0$: just take the downward closure of the given antichain. We improve this upper bound by showing that $\mathsf{WKL}_0$ suffices. We first point out that $\mathsf{RCA}_0$ proves a particular instance of the statement.

**Lemma 2.4.2** ($\mathsf{RCA}_0$). *Let $P$ be a partial order with a maximal (with respect to inclusion) infinite antichain. Then there exists an initial interval that is not a finite union of ideals.*

*Proof.* Let $D$ be a maximal infinite antichain of $P$. The maximality of $D$ implies that for all $x \in P$ we have

$$(\exists d \in D) x \preceq d \iff \neg(\exists d \in D) d \prec x.$$

Therefore the downward closure of $D$ is $\mathbf{\Delta}^0_1$ definable and thus exists in $\mathsf{RCA}_0$. Letting $I = \{x \in P \colon (\exists d \in D) x \preceq d\}$, we obtain an initial interval which is not finite union of ideals, since distinct elements of $D$ are incompatible in $I$. $\square$

To use Lemma 2.4.2 in the general case we need to extend an infinite antichain to a maximal one. While it is easy to show that $\mathsf{RCA}_0$ proves the existence of maximal antichains in any partial order, the statement that every antichain is contained in a maximal antichain is equivalent to $\mathsf{ACA}_0$, and thus does not help in our case.

**Lemma 2.4.3.** *Over $\mathsf{RCA}_0$, the following are equivalent:*

(1) $\mathsf{ACA}_0$;

(2) *every antichain in a partial order extends to a maximal antichain.*

*Proof.* We first show $(1) \Rightarrow (2)$. Let $P$ be a partial order and $D \subseteq P$ be an antichain. By recursion, we define $f \colon \mathbb{N} \to \{0, 1\}$ by letting $f(x) = 0$ if and only if either $x \notin P$ or

$x$ is comparable with some element of $(D \setminus \{x\}) \cup \{y \in P \colon y < x \wedge f(y) = 1\}$. Then $E = \{x \colon f(x) = 1\}$ is a maximal antichain with $D \subseteq E$.

For the reversal argue in $\mathsf{RCA}_0$ and fix a one-to-one function $f \colon \mathbb{N} \to \mathbb{N}$. Let $P = \{a_n, b_n \colon n \in \mathbb{N}\}$ and define the partial order by letting $b_m \preceq a_n$ if and only if $f(m) = n$, and adding no other comparabilities. Then apply $(2)$ to the antichain $D = \{b_m \colon m \in \mathbb{N}\}$ and obtain a maximal antichain $E$ such that $D \subseteq E$. It is immediate that $(\exists m) f(m) = n$ if and only if $a_n \notin E$, so that in $\mathsf{RCA}_0$ we can prove the existence of the range of $f$. □

We now show how to prove the right to left direction of Theorem 2.1.1 in $\mathsf{WKL}_0$.

**Theorem 2.4.4** ($\mathsf{WKL}_0$)**.** *Every partial order with an infinite antichain contains an initial interval that cannot be written as a finite union of ideals.*

*Proof.* We reason in $\mathsf{WKL}_0$. Let $P$ be a partial order such that every initial interval is a finite union of ideals. Suppose towards a contradiction that there exists an infinite antichain $D$.

Let $\varphi(x)$ and $\psi(x)$ be the $\mathbf{\Sigma}^0_1$ formulas $x \in D$ and $(\exists y)(y \in D \wedge y \prec x)$ respectively. It is obvious that $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \npreceq x)$. By $\mathbf{\Sigma}^0_1$ initial interval separation (Lemma 2.2.4), there exists an initial interval $I \subseteq P$ such that

$$(\forall x \in P)((\varphi(x) \implies x \in I) \wedge (\psi(x) \implies x \notin I)).$$

Therefore, $I$ contains $D$ and no element above any element of $D$. To see that $I$ cannot be the union of finitely many ideals notice that distinct $x, x' \in D$ cannot belong to the same ideal $A \subseteq I$, for otherwise there would be $z \in I$ such that $x, x' \preceq z$, which implies $\psi(z)$. □

We do not know whether the statement of Theorem 2.4.4 implies $\mathsf{WKL}_0$. However, the proof above uses the existence of an initial interval $I$ containing the infinite antichain $D$ and no elements above any element of $D$. We now show that even the existence of an initial interval $I$ containing infinitely many elements of the antichain $D$ and no elements above any element of $D$ is equivalent to $\mathsf{WKL}_0$. Therefore a proof of the right to left direction of Theorem 2.1.1 in a system weaker than $\mathsf{WKL}_0$ must avoid using such an $I$.

**Lemma 2.4.5.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{WKL}_0$;

(2) *if a partial order $P$ contains an infinite antichain $D$, then $P$ has an initial interval $I$ such that $D \subseteq I$ and $(\forall x \in D)(\forall y \in I) x \not\preceq y$;*

(3) *if a partial order $P$ contains an infinite antichain $D$, then $P$ has an initial interval $I$ such that $I \cap D$ is infinite and $(\forall x \in D)(\forall y \in I) x \not\preceq y$.*

*Proof.* The proof of $(1) \Rightarrow (2)$ is contained in Theorem 2.4.4 and $(2) \Rightarrow (3)$ is obvious, so that we just need to show $(3) \Rightarrow (1)$. Fix one-to-one functions $f, g \colon \mathbb{N} \to \mathbb{N}$ such that $(\forall n, m \in \mathbb{N}) f(n) \neq g(m)$. Let $P = \{a_n, b_n \colon n \in \mathbb{N}\}$ the partial order defined by letting

- $a_n \preceq b_m$ if $m = g(n)$;
- $b_k \preceq a_n$ if $(\exists i < n)(i < g(n) \wedge f(i) = k)$, i.e. $k$ enters the range of $f$ before stage $\min\{n, g(n)\}$;
- $b_k \preceq b_m$ if $(\exists i < m)(f(i) = k \wedge (\forall j < i) f(j) \neq m)$, i.e. $k$ enters the range of $f$ before stage $m$ and when $m$ has not entered the range of $f$ yet,

and adding no other comparabilities.

To check that $P$ is indeed a partial order we need to show that it is transitive. The main cases are the following:

- If $b_k \preceq a_n \preceq b_m$ we have $m = g(n)$ and the existence of $i < \min\{n, m\}$ such that $f(i) = k$. By the hypothesis on $f$ and $g$ we have $f(j) \neq m$ for every $j$, and in particular for every $j < i$, so that $b_k \preceq b_m$ follows.
- If $b_k \preceq b_m \preceq b_\ell$ there exist $i < m$ and $i' < \ell$ such that $f(i) = k$, $(\forall j < i) f(j) \neq m$, $f(i') = m$, and $(\forall j < i') f(j) \neq \ell$. The second and third condition imply $i \leq i'$, so that $i < \ell$, $(\forall j < i) f(j) \neq \ell$ and we obtain $b_k \preceq b_\ell$.
- If $b_k \preceq b_m \preceq a_n$ there exist $i < m$ and $i' < n$ such that $f(i) = k$, $(\forall j < i) f(j) \neq m$, $i' < g(n)$, and $f(i') = m$. Again we obtain $i \leq i'$, so that $i < \min\{n, g(n)\}$ and we can conclude $b_k \preceq a_n$.

The set $D = \{a_n \colon n \in \mathbb{N}\}$ is an infinite antichain. Applying (3) we obtain an initial interval $I$ of $P$ which contains infinitely many elements of $D$ and no elements above any element of $D$. We now check that $\{k \in \mathbb{N} \colon b_k \in I\}$ separates the range of $f$ from the range of $g$.

If $k = g(n)$ it is immediate that $a_n \prec b_k$ so that $b_k \notin I$.

On the other hand suppose that $k = f(i)$. The set $A = \{n \colon g(n) \leq i\}$ is finite by the injectivity of $g$ and we can let $m = \max(\{i\} \cup A)$. Since $D \cap I$ is infinite there exists $n > m$ such that $a_n \in I$. Then we have $i < n$ and $i < g(n)$ (because $n \notin A$), so that $b_k \preceq a_n$. Therefore $b_k \in I$. □

We notice that another weakening of statement $(2)$ of Lemma 2.4.5 which is equivalent to $\mathsf{WKL}_0$ is the following: "if a partial order $P$ contains an infinite antichain $D$, then there exists an initial interval $I$ such that $D \subseteq I$ and $(\forall y \in I)(\exists^\infty x \in D) x \not\preceq y$" (the proof of the reversal uses the partial order of the proof above equipped with the inverse order). However this statement does not imply the statement of Theorem 2.4.4.

## 2.5 Unprovability in $\mathsf{RCA}_0$

In this section we show that $\mathsf{RCA}_0$ does not suffice to prove the right to left direction of Theorem 2.1.1.

**Lemma 2.5.1.** *There exists a computable partial order $P$ with an infinite computable antichain such that any computable initial interval of $P$ is the downward closure of a finite subset of $P$.*

Before proving Lemma 2.5.1 we show how to deduce from it the unprovability result.

**Theorem 2.5.2.** $\mathsf{RCA}_0$ *does not prove that every partial order such that all its initial intervals are finite union of ideals is FAC.*

*Proof.* It suffices to show that the statement fails in $\mathbf{REC}$, the $\omega$-model of computable sets. Let $P$ the computable partial order of Lemma 2.5.1 and let $I$ be a computable initial interval of $P$. Let $F$ be a finite set such that $P = {\downarrow} F$. Then $I = \bigcup_{x \in F} P(\preceq x)$ and each $P(\preceq x)$ is a computable ideal.

Thus all initial intervals of $P$ which belong to $\mathbf{REC}$ are finite union of ideals also belonging to $\mathbf{REC}$. On the other hand, $P$ has an infinite antichain in $\mathbf{REC}$, showing the failure of the statement. □

*Proof of Lemma 2.5.1.* We build $P$ by a finite injury priority argument. We let $P = \{x_n, y_n \colon n \in \omega\}$ and ensure the existence of an infinite computable antichain by making the $x_n$'s pairwise incomparable.

We further make sure that, for all $e \in \omega$, $P$ meets the requirement:

$$R_e\colon (\exists y)\big((\Phi_e(y) = 1 \implies (\forall^\infty z \in P) z \preceq y) \wedge (\Phi_e(y) = 0 \implies (\forall^\infty z \in P) y \preceq z)\big).$$

Here, as usual, $\Phi_e$ is the function computed by the Turing machine of index $e$ and $\forall^\infty$ means 'for all but finitely many'.

We first show that meeting all the requirements implies that $P$ satisfies the statement of the Lemma. If $I$ is a computable initial interval of $P$ with characteristic function $\Phi_e$, fix $y$ given by $R_e$. We must have $\Phi_e(y) \in \{0, 1\}$. If $\Phi_e(y) = 0$ then, by $R_e$, $(\forall^\infty z \in P) y \preceq z$. As $y \notin I$, this implies that $I$ is finite and hence $I = {\downarrow} I$ is the downward closure of a finite set. If $\Phi_e(y) = 1$, then by $R_e$ we have $(\forall^\infty z \in P) z \preceq y$. Thus $P \setminus P(\preceq y)$ and hence $I \setminus P(\preceq y)$ are finite. As $y \in I$, $I = {\downarrow}(\{y\} \cup (I \setminus P(\preceq y)))$ is the downward closure of a finite set.

Our strategy for meeting a single requirement $R_e$ consists in fixing a witness $y_n$ and waiting for a stage $s + 1$ such that

$$\Phi_{e,s}(y_n) \in \{0, 1\}.$$

If this never happens, $R_e$ is satisfied. If $\Phi_{e,s}(y_n) = 0$, we put every $x_m$ and $y_m$ with $m > s$ above $y_n$. If $\Phi_{e,s}(y_n) = 1$, we put every $x_m$ and $y_m$ with $m > s$ below $y_n$. In this way $R_e$ is obviously satisfied.

To meet all the requirements, the priority order is $R_0, R_1, R_2, \ldots$. At every stage $s$, we define a witness for $R_e$ via an index $n_{e,s}$ and mark the requirements by a $\{0, 1\}$-valued function $r(e, s)$ such that $r(e, s) = 0$ if and only if $R_e$ might require attention at stage $s$.

**Construction.**

Stage $s = 0$. For all $e$, $n_{e,0} = e$ and $r(e, 0) = 0$.

Stage $s + 1$. We say that $R_e$ *requires attention* at stage $s + 1$ if $e \leq s$, $n_{e,s} \leq s$, $r(e, s) = 0$ and $\Phi_{e,s}(y_{n_{e,s}}) \in \{0, 1\}$. If no $R_e$ requires attention, then let $n_{i,s+1} = n_{i,s}$ and $r(i, s + 1) = r(i, s)$ for all $i$. Otherwise, let $e$ be least such that $R_e$ requires attention. Then $R_e$ *receives attention* at stage $s + 1$ and $n = n_{e,s}$ is *activated* and declared *low* if $\Phi_{e,s}(y_n) = 0$, ***high*** if $\Phi_{e,s}(y_n) = 1$. Let $n_{e,s+1} = n_{e,s}$ and $r(e, s + 1) = 1$. For $i < e$, $n_{i,s+1} = n_{i,s}$ and $r(i, s + 1) = r(i, s)$. For $i > e$, $n_{i,s+1} = s + i - e$ and $r(i, s + 1) = 0$.

The following two properties are easily seen to hold:

1. every $n$ is activated at most once;

2. if $n$ is activated at stage $s$, then no $m$ such that $n < m < s$ is activated after $s$.

We define $\preceq$ by stipulating that for all $n < m$:

(i) $x_n$ is incomparable with each of $y_n$, $x_m$ and $y_m$;

(ii) $y_n \preceq (\succeq)\, x_m, y_m$ if and only if $n$ is activated at some stage $s$ such that $n < s \leq m$, is declared low (high) and no $k < n$ is activated at any stage $t$ such that $s < t \leq m$.

When (ii) occurs, it follows by (2) that no $k < n$ is activated at any stage $t$ such that $n < t \leq m$.

**Claim 1.** $P$ is a partial order.

*Proof of claim.* We use $z_n$ to denote one of $x_n$ and $y_n$.

To show antisymmetry, suppose for a contradiction that $z_n \preceq z_m$ and $z_m \preceq z_n$ with $n < m$. By (i) $z_n$ must be $y_n$. Since $n$ can be activated only once, it follows that $n$ is activated at some stage $s$ with $n < s \leq m$ and, by (ii), is declared both low and high, a contradiction.

To check transitivity, let $z_n \prec z_m \prec z_p$. Notice that $n$, $m$ and $p$ are all distinct. We consider the following cases:

- $n < m, p$. Then $z_n = y_n$ and $n$ is activated and declared low at some stage $s$ such that $n < s \leq m$. It is easy to verify that no $k < n$ is activated at any stage $t$ such that $n < t \leq p$, and thus $y_n \preceq z_p$.
- $m < n, p$. Then $z_m = y_m$ and $m$ is declared both high and low, contradiction.
- $p < n, m$. Then $z_p = y_p$ and $p$ is activated and declared high at some stage $s$ such that $p < s \leq m$. As in case (a), it is easy to check that no $k < p$ is activated at any stage $t$ such that $p < t \leq n$, and so $z_n \preceq y_p$. □

**Claim 2.** Every $R_e$ receives attention at most finitely often and is satisfied.

*Proof of claim.* As usual, the proof is by induction on $e$. Let $s$ be the least such that no $R_i$ with $i < e$ receives attention after $s$. Let $n = n_{e,s}$. Then $n = n_{e,t}$ for all $t \geq s$, because a witness for a requirement changes only when a stronger priority requirement receives attention. Similarly, $r(e,t) = 0$ for all $t \geq s$ such that $R_e$ has not received attention at any stage between $s$ and $t$. If $\Phi_e(y_n) \notin \{0,1\}$, $R_e$ is clearly satisfied. Suppose that $\Phi_e(y_n) = 0$

(case 1 is similar) and let $t$ be minimal such that $t \geq \max\{s, e, n\}$ and $\Phi_{e,t}(y_n) = 0$. Then $R_e$ receives attention at stage $t+1$, $n$ is activated and declared low and no $m < n$ will be activated after stage $t+1$ (because $n_{i,u} > n$ for all $i > e$ and $u > t$). Then $y_n \preceq x_m, y_m$ for all $m > t$ and so $R_e$ is satisfied. □

Claim 2 completes the proof of the Lemma. □

### 2.5.1 Other unprovability results

Ludovic Patey [BPS] improved our result and showed that $\mathsf{WWKL}_0$ does not imply the statement of Theorem 2.4.4 by modifying the proof of Lemma 2.5.1 and proving:

**Theorem 2.5.3** (Patey)**.** *There exists a computable partial order $P$ with an infinite computable antichain such that the set of reals computing an initial interval which is not the downward closure of a finite set is null.*

**Corollary 2.5.4.** $\mathsf{WWKL}_0$ *does not prove that that every partial order such that its initial intervals are finite union of ideals is FAC.*

*Proof.* It follows by Theorem 1.4.5 and Theorem 2.5.3 that there exists a Martin-Löf random real $X$ such that any initial interval of $P$ computed by $X$ is the downward closure of a finite set. On the other hand, by Theorem 1.4.4, there exists an $\omega$-model $M$ of $\mathsf{WWKL}_0$ such that any set in the model is computable in $X$. Since $M \supseteq \mathbf{REC}$, $P \in M$ and $P$ is not FAC in $M$. The argument showing that in $M$ every initial interval of $P$ is a finite union of ideals is the same as in the proof of Theorem 2.5.2. □

We do not know whether $\mathsf{RT}^2_2$ implies the right to left direction of Theorem 2.1.1. However, by the following conservation result, we obtain that $\mathsf{COH}$ (*Cohesive Principle*), which is a consequence of $\mathsf{RT}^2_2$, cannot imply it.

**Theorem 2.5.5** ([HS07])**.** $\mathsf{COH}$ *is conservative over* $\mathsf{RCA}_0$ *for $\mathbf{\Pi}^1_2$ statements of the form $(\forall X)(\theta(X) \implies (\exists Y)\varphi(X, Y))$, where $\theta$ is arithmetical and $\varphi$ is $\mathbf{\Sigma}^0_3$.*

**Corollary 2.5.6.** *Over* $\mathsf{RCA}_0$*,* $\mathsf{COH}$ *does not imply the statement that every partial order such that all its initial intervals are finite union of ideals is FAC.*

*Proof.* We cannot apply directly the above conservation result, because our statement is indeed $\mathbf{\Pi}^1_3$. However, it follows by Lemma 2.5.1 that $\mathsf{RCA}_0$ does not even prove the weaker $\mathbf{\Pi}^1_2$ statement "every partial order containing an infinite antichain has an initial interval which is not the downward closure of a finite set". The latter is of the form required and hence COH does not prove it. Therefore, COH does not prove our statement either. $\square$

# 3

# Well-scattered partial orders and Erdös-Rado theorem

## 3.1 Introduction

We consider a characterization theorem for scattered FAC partial orders (Theorem 3.1.2) analogous to that for well-partial orders (Theorem 3.1.3). For the sake of analogy and for notational convenience, we give the following definition.

**Definition 3.1.1.** A partial order $P$ is a *well-scattered partial order* ($\mathsf{wspo}$) if for every function $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$.

The theorem below provides four classically equivalent definitions for well-scattered partial orders.

**Theorem 3.1.2** ([BP69])**.** *Let $P$ be a partial order. The following are equivalent:*

$\mathsf{wspo(ant)}$ *$P$ is scattered and FAC;*

$\mathsf{wspo(ext)}$ *every linear extension of $P$ is scattered;*

$\mathsf{wspo}$ *$P$ is a well-scattered partial order;*

$\mathsf{wspo(set)}$ *for every function $f\colon \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

We aim to study the reverse mathematics of Theorem 3.1.2. To do this, we consider one statement for every pair of equivalent conditions. For instance, $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$ denotes the statement "for every partial order $P$, if $P$ is scattered and FAC, then every linear extension of $P$ is scattered".

As said before, we have similar conditions for well-partial orders.

**Theorem 3.1.3.** *Let $P$ be a partial order. Then the following are equivalent:*

wpo(ant) *$P$ is well-founded and has no infinite antichains;*

wpo(ext) *every linear extension of $P$ is well-founded;*

wpo *$P$ is a well-partial order, i.e. for every function $f\colon \mathbb{N} \to P$ there exist $x < y$ such that $f(x) \leq_P f(y)$;*

wpo(set) *for every function $f\colon \mathbb{N} \to P$ there exist an infinite set $A \subseteq \mathbb{N}$ such that $x < y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

The reverse mathematics of Theorem 3.1.3 was studied in [CMS04] (see Table 1). Here we strengthen a few results by showing that most statements are equivalent to CAC (see Table 2).

| $\to$ | wpo(ant) | wpo(ext) | wpo | wpo(set) |
|---|---|---|---|---|
| wpo(ant) | | CAC $\Rightarrow$<br>**REC** $\not\models$<br>[CMS04, 3.10]<br>$\mathsf{WKL}_0 \nvdash$<br>[CMS04, 3.19] | CAC $\Rightarrow$<br>**REC** $\not\models$<br>[CMS04, 3.9]<br>$\mathsf{WKL}_0 \nvdash$<br>[CMS04, 3.11] | $\Leftrightarrow$ CAC<br>[CMS04, 3.3]<br>$\mathsf{WKL}_0 \nvdash$ |
| wpo(ext) | $\mathsf{WKL}_0 \vdash$<br>**REC** $\not\models$<br>[CMS04, 3.21] | | $\mathsf{WKL}_0 \vdash$<br>[CMS04, 3.17]<br>**REC** $\not\models$<br>[CMS04, 3.21] | $\Rightarrow \mathsf{RT}^1_{<\infty}$<br>$\mathsf{WKL}_0 + \mathsf{CAC} \vdash$<br>**REC** $\not\models$<br>$\mathsf{WKL}_0 \nvdash$ |
| wpo | | | | CAC $\Rightarrow$<br>$\Rightarrow \mathsf{RT}^1_{<\infty}$<br>[CMS04, 2.5]<br>**REC** $\not\models$<br>[CMS04, 3.7]<br>$\mathsf{WKL}_0 \nvdash$ |
| wpo(set) | | | | |

Table 1: Known results for well-partial orders[1]

[1]The entry row$\to$column contains the results for the corresponding statement, while an empty entry stands for a statement provable in $\mathsf{RCA}_0$.

| $\to$ | wpo(ant) | wpo(ext) | wpo | wpo(set) |
|---|---|---|---|---|
| wpo(ant) | | $\Leftrightarrow$ CAC (3.5.4) | $\Leftrightarrow$ CAC (3.5.4) | |
| wpo(ext) | WWKL$_0$ $\nvdash$ (3.6.4) | | | $\Rightarrow$ CAC |
| wpo | | | | $\Leftrightarrow$ CAC (3.5.5) |

Table 2: New results for well-partial orders

As for well-scattered partial orders, it turns out that a partition theorem for rationals that we call $\mathsf{ER}^2_2$ (after Erdös-Rado) plays the role of $\mathsf{RT}^2_2$ in the reverse mathematics of Theorem 3.1.2.

**Theorem 3.1.4** ([ER52], Theorem 4, p. 427)**.** *The partition relation* $\mathbb{Q} \to (\aleph_0, \mathbb{Q})^2$ *holds.*

The theorem says that for every coloring $c \colon [\mathbb{Q}]^2 \to 2$ there exists either an infinite $0$-homogeneous set or a dense $1$-homogeneous set.

Actually, we shall consider semitransitive versions of $\mathsf{ER}^2_2$ (namely st-$\mathsf{ER}^2_2$ and st-$\mathsf{ER}^2_3$). Table 3 below contains our results.

| $\to$ | wspo(ant) | wspo(ext) | wspo | wspo(set) |
|---|---|---|---|---|
| wspo(ant) | | $\Leftrightarrow$ st-ER$^2_2$ (3.5.9, 3.5.10)<br>**REC** $\not\models$ (3.6.6)<br>$\Rightarrow$ RT$^1_{<\infty}$<br>WKL$_0$ $\nvdash$ | st-ER$^2_3$ $\Rightarrow$<br>$\Rightarrow$ st-ER$^2_2$<br>**REC** $\not\models$<br>$\Rightarrow$ RT$^1_{<\infty}$<br>WKL$_0$ $\nvdash$ | $\Leftrightarrow$ st-ER$^2_3$ (3.5.10)<br>$\Rightarrow$ st-RT$^2_2$<br>**REC** $\not\models$<br>$\Rightarrow$ RT$^1_{<\infty}$<br>WKL$_0$ $\nvdash$ |
| wspo(ext) | WKL$_0$ $\vdash$<br>**REC** $\not\models$ (3.6.2)<br>WWKL$_0$ $\nvdash$ (3.6.5) | | WKL$_0$ $\vdash$ (3.4.4)<br>**REC** $\not\models$<br>WWKL$_0$ $\nvdash$ | $\Rightarrow$ st-RT$^2_2$<br>WKL$_0$ + st-ER$^2_3$ $\vdash$ |
| wspo | | | | st-ER$^2_3$ $\Rightarrow$<br>$\Rightarrow$ st-RT$^2_2$ (3.5.9) |
| wspo(set) | | | | |

Table 3: Results for well-scattered partial orders

## 3.2 Erdös-Rado partition relation

In this section we focus on the proof of Erdös-Rado theorem and we draw some computability-theoretic and reverse mathematics consequences. Recall the statement:

**Theorem 3.2.1.** *For every coloring* $c\colon [\mathbb{Q}]^2 \to 2$ *there exists either an infinite* $0$*-homogeneous set or a dense* $1$*-homogeneous set.*

**Definition 3.2.2.** We say that $A \subseteq \mathbb{Q}$ is *locally dense* if $A$ is dense in some open interval of $\mathbb{Q}$ (i.e. $A$ is locally dense in the order topology of the rationals).

Notice that if $A \cup B$ is locally dense then either $A$ or $B$ is locally dense. Thus, the collection of non-locally dense subsets of $\mathbb{Q}$ is an ideal on $\mathcal{P}(\omega)$. Therefore we call *positive* any locally-dense set and *small* any non-positive set.

*Proof of Theorem 3.2.1.* Let $c\colon [\mathbb{Q}]^2 \to 2$ be given. For any $x \in \mathbb{Q}$, let $R(x) = \{y \in \mathbb{Q} \setminus \{x\}\colon c(x, y) = 0\}$ and $B(x) = \{y \in \mathbb{Q} \setminus \{x\}\colon c(x, y) = 1\}$. A subset $A \subseteq \mathbb{Q}$ is said to be *red-admissible* if there exists $x \in A$ such that $A \cap R(x)$ is positive

Case $(1)$. Every positive subset of $\mathbb{Q}$ is red-admissible. Let $A_0 = \mathbb{Q}$. Clearly, $A_0$ positive. Then, by hypothesis, $A_0$ is red-admissible and hence there exists $x_0 \in A_0$ such that $A_1 = A_0 \cap R(x_0) = R(x_0)$ is positive. Then $A_1$ is red-admissible and so there exists $x_1 \in A_1$ such that $A_2 = A_1 \cap R(x_1) = R(x_0) \cap R(x_1)$ is positive. Suppose we have defined $A_n = \bigcap_{k<n} R(x_k)$ such that $A_n$ is positive. Then there exists $x_n \in A_n$ such that $A_{n+1} = A_n \cap R(x_n)$ is positive. Therefore, for all $n$, $x_n \in A_n = \bigcap_{k<n} R(x_k)$. Hence, $\{x_n\colon n \in \mathbb{N}\}$ is an infinite $0$-homogeneous subset of $\mathbb{Q}$.

Case $(2)$. There is a positive subset $A$ of $\mathbb{Q}$ which is not red-admissible. Suppose $A$ is dense in the open interval $I$. Fix an enumeration $(I_n)$ of all open intervals contained in $I$. Notice that $A$ intersects every $I_n$.

Let $x_0 \in A \cap I_0$. Suppose we have defined $x_k \in A \cap I_k$ for all $k < n$. Since none of the sets $A \cap R(x_k)$ is positive, it follows that $\bigcup_{k<n} A \cap R(x_k) = A \cap \bigcup_{k<n} R(x_k)$ is small. Let $J \subseteq I_n$ be such that $J \cap A \cap \bigcup_{k<n} R(x_k) \cup \{x_k\} = \emptyset$. Since $A$ is dense in $I$, we can find $x_n \in A \cap J$. It follows that $x_n \in \bigcap_{k<n} B(x_k)$. Therefore, for all $n$, $x_n \in I_n \cap \bigcap_{k<n} B(x_k)$. Hence, $\{x_n\colon n \in \mathbb{N}\}$ is a dense $1$-homogeneous subset of $\mathbb{Q}$. $\square$

A straightforward effectivization of the proof of Theorem 3.2.1 yields the following:

**Lemma 3.2.3.** *Every computable coloring of pairs of rationals has a* $\mathbf{\Delta}^0_3$ *solution.*

*Proof.* Let $c\colon [\mathbb{Q}]^2 \to 2$ be a computable coloring. (Here, $\mathbb{Q}$ is a computable presentation of the rationals on $\omega$.)

Case $(1)$. Every computable positive set is red-admissible. Computably enumerate all pairs $(x, I)$, where $x \in \mathbb{Q}$ and $I$ is an open interval of $\mathbb{Q}$. Look for $(x_0, I_0)$ such that $R(x_0)$ is dense in $I_0$. We can ask $0''$ whether $R(x_0)$ is dense in $I_0$, the question being $\mathbf{\Sigma}^0_2$. Keep on looking for pairs $(x_n, I_n)$ such that $x_n \in \bigcap_{k<n} R(x_k)$ and $\bigcap_{k<n+1} R(x_k)$ is dense in $I_n$ for all $n$. We can make the infinite set $\{x_n : n \in \omega\}$ computable in $0''$ by searching for pairs $(x_n, I_n)$ with $x_n < x_{n+1}$. Therefore, we have a $\mathbf{\Delta}^0_3$ infinite $0$-homogeneous set.

Case $(2)$. There exists a computable positive set $A$ that is not red-admissible. Fix an open interval $I$ such that $A$ is dense in $I$ and computably enumerate all open intervals contained in $I$. The proof of the theorem shows that we can find $x_n \in A \cap I_n \cap \bigcap_{k<n} B(x_k)$ for all $n$. The search is computable in $A$ and the enumeration. Therefore, we have a computable dense $1$-homogeneous set. □

Denote by $\mathsf{ER}^2_2$ the statement of Theorem 3.2.1.

**Lemma 3.2.4.** $\mathsf{ER}^2_2$ *is provable in* $\mathsf{ACA}_0$ *and implies* $\mathsf{RT}^2_2$ *over* $\mathsf{RCA}_0$.

*Proof.* The proof of Theorem 3.2.1 above can be formalized in $\mathsf{ACA}_0$. The second implication is trivial (order the natural numbers like $\mathbb{Q}$, apply $\mathsf{ER}^2_2$ and forget the order). □

## 3.3 $\Sigma^0_1$ dense chains

It is well-known in computability theory that an infinite $\mathbf{\Sigma}^0_1$ set contains a computable infinite subset. This is provable in $\mathsf{RCA}_0$ and hence any partial order containing an infinite $\mathbf{\Sigma}^0_1$ chain (or antichain) contains an infinite $\mathbf{\Delta}^0_1$ chain (or antichain). We next show that the same holds for dense chains.

**Lemma 3.3.1** ($\mathsf{RCA}_0$)**.** *Let* $f\colon \mathbb{Q} \to \mathbb{N}$ *be a one-to-one function. Then there exists an embedding* $g\colon \mathbb{Q} \to \mathbb{Q}$ *such that* $\operatorname{ran}(f \circ g)$ *exists.*

*Proof.* Let $f\colon \mathbb{Q} \to \mathbb{N}$ be as above. We then define $g\colon \mathbb{Q} \to \mathbb{Q}$ by recursion. Suppose we have defined $g(k)$ for all $k < n$. We assume by $\mathbf{\Sigma}^0_0$ induction that for all $j, k < n$

$$j <_{\mathbb{Q}} k \implies g(j) <_{\mathbb{Q}} g(k), \text{ and}$$

$$j < k \implies f(g(j)) < f(g(k)).$$

Search for the least $m \in \mathbb{N}$ such that for all $k < n$

$$k <_{\mathbb{Q}} n \text{ if and only if } g(k) <_{\mathbb{Q}} m \tag{$*$}$$

and $f(g(k)) < f(m)$. Since there are infinitely many $m$ such that $(*)$ holds and $f$ is one-to-one, the search will succeed. Then let $g(n) = m$.

The function $g$ so defined is clearly an embedding from $\mathbb{Q}$ to itself. Also, $\operatorname{ran}(f \circ g)$ is $\boldsymbol{\Delta}^0_1$ definable and so exists in $\mathsf{RCA}_0$. □

**Corollary 3.3.2** ($\mathsf{RCA}_0$). *A partial order is scattered if and only if it does not contain any dense subchain.*

*Proof.* The left to right direction is immediate because $\mathsf{RCA}_0$ suffices to carry out the usual back-and-forth argument. For the other direction, if $f\colon \mathbb{Q} \to P$ is an embedding, $f$ is one-to-one and by Lemma 3.3.1 there is an embedding $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $D = \operatorname{ran}(f \circ g)$ exists. Therefore $D$ is the range of an embedding of $\mathbb{Q}$ into $P$ and so is a dense subchain of $P$. □

As a result, we also obtain the following two results:

**Corollary 3.3.3** ($\mathsf{RCA}_0$). *Let $P$ be a partial order. Then the following are equivalent:*

(1) *for every $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$ (i.e.* $\mathsf{wspo}$*);*

(2) *every restriction of $P$ has no dense linear extensions.*

*Proof.* (1) $\Rightarrow$ (2). We prove the contrapositive. Let $X \subseteq P$ and suppose that $L$ is a dense linear extension of $X$. By Corollary 3.3.2, there exists an embedding $f\colon \mathbb{Q} \to L$. It is easy to check that $f$ contradicts $\mathsf{wspo}$.

(2) $\Rightarrow$ (1). Once again we prove the contrapositive. Let $f\colon \mathbb{Q} \to P$ be such that $f(x) \not\leq_P f(y)$ for all $x <_{\mathbb{Q}} y$. In particular, $f$ is one-to-one, and hence satisfies the hypothesis of Lemma 3.3.1. We thus may assume that $\operatorname{ran}(f)$ exists. Let $X = \operatorname{ran}(f)$ and define a dense linear extension $L$ of $X$ by letting $x <_L y$ if and only if $f^{-1}(x) >_{\mathbb{Q}} f^{-1}(y)$. □

**Corollary 3.3.4.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) *every linear extension of a scattered FAC partial order is scattered (i.e.* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$*);*

(2) *every linear extension of a scattered FAC partial order is not dense.*

*Proof.* Straightforward. □

## 3.4 Proof-theoretic upper bounds

**Lemma 3.4.1.** $\mathsf{RCA}_0$ *proves:*

(1) $\mathsf{wspo(set)} \to \mathsf{wspo}$*;*

(2) $\mathsf{wspo} \to \mathsf{wspo(ant)}$*;*

(3) $\mathsf{wspo} \to \mathsf{wspo(ext)}$.

*Proof.* (1) is trivial. For (2) and (3), let us consider the contrapositives. If $P$ contains an infinite antichain $A$, let $f$ be any one-to-one function from $\mathbb{Q}$ to $A$. If $P$ is non-scattered, let $f$ be an embedding of $\mathbb{Q}^*$ into $P$. Finally, if $P$ has a non-scattered linear extension $L$, let $f$ be an embedding of $\mathbb{Q}^*$ into $L$. In either case $f$ contradicts $\mathsf{wspo}$. □

### Fact provable in $\mathsf{WKL}_0$

**Lemma 3.4.2** ([CMS04])**.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{WKL}_0$*;*

(2) *every acyclic relation can be extended to a partial order.*

**Corollary 3.4.3.** $\mathsf{WKL}_0$ *proves that every acyclic relation can be extended to a linear order.*

*Proof.* $\mathsf{RCA}_0$ suffices to prove that every partial order has a linear extension. □

**Theorem 3.4.4.** $\mathsf{WKL}_0$ *proves* $\mathsf{wspo(ext)} \to \mathsf{wspo}$.

*Proof.* Let $P$ be a partial order such that any linear extension of $P$ is scattered. Suppose for a contradiction that there is a function $f\colon \mathbb{Q} \to P$ such that

$$x <_{\mathbb{Q}} y \text{ implies } f(x) \perp f(y) \text{ or } f(x) >_P f(y) \text{ for all } x, y \in \mathbb{Q}. \qquad (*)$$

In particular, $f$ is injective and by Lemma 3.3.1 we may assume that $\operatorname{ran}(f)$ exists. We thus define a binary relation $\mathrm{R} \subseteq P^2$ by letting $u \,\mathrm{R}\, v$ if and only if

$$u >_P v \text{ or } x = f(u) \wedge y = f(v) \text{ for some } x <_{\mathbb{Q}} y.$$

We claim that $\mathrm{R}$ is acyclic, i.e. there is no sequence $u_0 \,\mathrm{R}\, u_1 \,\mathrm{R}\, u_2 \ldots \mathrm{R}\, u_n$ with $u_n \,\mathrm{R}\, u_0$. We show this by $\mathbf{\Pi}^0_1$ induction on $n \in \mathbb{N}$.

If $n = 1$, since $\leq_P$ is antisymmetric, we may assume $u_0 = f(x)$ and $u_1 = f(y)$ for some $x, y \in \mathbb{Q}$. Now, $u_0 \,\mathrm{R}\, u_1$ implies $x <_{\mathbb{Q}} y$ and $u_1 \,\mathrm{R}\, u_0$ implies $x >_{\mathbb{Q}} y$, a contradiction. Let $n > 1$ and set $u_{-1} = u_n$ and $u_{n+1} = u_0$. If $u_k \notin \operatorname{ran}(f)$ for some $0 \leq k \leq n$, then $u_{k-1} >_P u_{k+1}$ and so $u_0 \,\mathrm{R} \ldots u_{k-1} \,\mathrm{R}\, u_{k+1} \,\mathrm{R} \ldots \mathrm{R}\, u_n$ is a cycle of length $n-1$ and the induction hypothesis applies. Otherwise, for all $0 \leq k \leq n$, let $x_k \in \mathbb{Q}$ be the unique $x \in \mathbb{Q}$ such that $f(x) = u_k$. Therefore $x_0 <_{\mathbb{Q}} \ldots <_{\mathbb{Q}} x_n$ and $x_n <_{\mathbb{Q}} x_0$, a contradiction again.

By Corollary 3.4.3, $\mathrm{R}$ extends to a linear order $L$. It is straightforward to verify that $L$ is an extension of $P$ and $f$ is an embedding of $\mathbb{Q}^*$ into $L$, contrary to assumption. $\square$

**Corollary 3.4.5.** $\mathsf{WKL}_0$ *proves* $\mathsf{wspo(ext)} \rightarrow \mathsf{wspo(ant)}$.

*Proof.* Immediate from Lemma 3.4.1. $\square$

## Facts provable in $\mathsf{ACA}_0$

**Theorem 3.4.6.** *Over* $\mathsf{RCA}_0$, $\mathsf{ER}^2_2$ *implies* $\mathsf{wspo(ant)} \rightarrow \mathsf{wspo(set)}$.

*Proof.* Let $P$ be a scattered FAC partial order and let $f \colon \mathbb{Q} \to P$. We aim to show that there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$. Let $c \colon [\mathbb{Q}]^2 \to 2$ defined by letting

$$c(x, y) := \begin{cases} 0 & \text{if } f(x) \perp_P f(y), \\ 1 & \text{otherwise.} \end{cases}$$

Apply $\mathsf{ER}^2_2$ to $c$. If $B \subseteq \mathbb{Q}$ is an infinite 0-homogeneous set, then the range of $f \restriction B$ is an infinite antichain. Since any $\mathbf{\Sigma}^0_1$ infinite antichain contains a $\mathbf{\Delta}^0_1$ infinite antichain, this contradicts $P$ FAC. Then there must exist a dense 1-homogeneous set $B$. Consider

$d\colon [B]^2 \to 2$ defined by letting

$$d(x, y) := \begin{cases} 0 & \text{if } x <_{\mathbb{Q}} y \Leftrightarrow f(x) \leq_P f(y), \\ 1 & \text{otherwise.} \end{cases}$$

Apply ${}^*\mathsf{ER}^2_2$ to $d$. If $A \subseteq B$ is a dense 1-homogeneous set, then $f \restriction A$ is an embedding of $A^*$ into $P$, contradicting $P$ scattered. It follows that there is an infinite 0-homogeneous set $A$ for $d$. Therefore $A$ is as desired. □

**Corollary 3.4.7.** *Over* $\mathsf{RCA}_0$, $\mathsf{ER}^2_2$ *implies* $\mathsf{wspo(ant)} \to \mathsf{wspo}$ *and* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$.

*Proof.* Immediate from Lemma 3.4.1. □

## 3.5 Semitransitive colorings

In [CMS04] it is shown that CAC is equivalent to $\mathsf{wpo(ant)} \to \mathsf{wpo(set)}$ (see table 1). In [HS07] the authors define the notion of *semitransitive* coloring and prove that CAC is equivalent to the semitransitive version of $\mathsf{RT}^2_2$. By using the latter result, we show that CAC is equivalent to other three statements involving well-partial orders.

**Definition 3.5.1** ([HS07], Definition 5.1)**.** A coloring $c\colon [\mathbb{N}]^2 \to n$ is transitive on $i < n$ if $c(x, y) = c(y, z) = i$ implies $c(x, z) = i$ for all $x < y < z$. A coloring $c\colon [\mathbb{N}]^2 \to n$ is *semitransitive* if it is transitive on every $i > 0$.

We also isolate the corresponding notion of *semitransitive* coloring for pairs of rationals and prove a few results about well-scattered partial orders.

The following generalization will apply to colorings of rationals.

**Definition 3.5.2.** Let $L$ be a linear order. A coloring $c\colon [L]^2 \to n$ is *transitive on* $i < n$ if $c(x, y) = c(y, z) = i$ implies $c(x, z) = i$ for all $x <_L y <_L z$. We say that $c$ is semitransitive if it is transitive on every $i > 0$.

### Application to well-partial orders

For all $n \geq 2$, let:

$\mathsf{st\text{-}RT}^2_n$: every semitransitive coloring $c\colon [\mathbb{N}]^2 \to n$ has an infinite homogeneous set.

**Theorem 3.5.3** ([HS07], Theorem 5.2)**.** *For all $n \geq 2$,* $\mathsf{RCA}_0$ *proves* $\mathsf{CAC} \Leftrightarrow \mathsf{st\text{-}RT}^2_n$.

**Theorem 3.5.4.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{CAC}$*;*

(2) $\mathsf{wpo(ant)} \to \mathsf{wpo}$*;*

(3) $\mathsf{wpo(ant)} \to \mathsf{wpo(ext)}$.

*Proof.* $(1) \Rightarrow (2)$ is [CMS04, Corollary 3.5] (see also table 1). $(2) \Rightarrow (3)$ is immediate since $\mathsf{wpo} \to \mathsf{wpo(ext)}$ is provable in $\mathsf{RCA}_0$.

We next show $(3) \Rightarrow (1)$. Assume $(3)$. By Theorem 3.5.3, it is enough to prove $\mathsf{st\text{-}RT}^2_2$. Let $c\colon [\mathbb{N}]^2 \to 2$ be a semitransitive coloring. By definition, $c$ is transitive on $1$ and so we can define a partial order $P$ by letting $x \leq_P y$ if and only if $x = y$ or $x > y$ and $c(x, y) = 1$. Since $x \leq_P y$ implies $x \geq y$, $\omega^*$ is a linear extension of $P$ and so $P$ has a non-well-founded linear extension. Now, an infinite antichain on $P$ is an infinite $0$-homogeneous set and an infinite descending sequence of $P$ yields an infinite $1$-homogeneous set. □

**Theorem 3.5.5.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{CAC}$*;*

(2) $\mathsf{wpo} \to \mathsf{wpo(set)}$.

*Proof.* $(1) \Rightarrow (2)$ is [CMS04, Corollary 3.4]. For the other direction, let us show $\mathsf{st\text{-}RT}^2_2$. Let $c\colon [\mathbb{N}]^2 \to 2$ be semitransitive. Define a partial order $P$ by letting $x \leq_P y$ if and only if $x = y$ or $x < y$ and $c(x, y) = 1$. Suppose first that $P$ is not $\mathsf{wpo}$ and let $f\colon \mathbb{N} \to P$ be a witness. This means that $x < y$ implies $f(x) \nleq_P f(y)$ for all $x, y \in \mathbb{N}$. We can assume without loss of generality that $f(x) < f(y)$ for all $x < y$. It follows that $x < y$ implies $c(f(x), f(y)) = 0$ and so the range of $f$, which exists by $\mathbf{\Delta}^0_1$ comprehension, is an infinite $0$-homogeneous set. Suppose instead that $P$ is $\mathsf{wpo}$. By $(2)$, $P$ satisfies $\mathsf{wpo(set)}$. Let $f\colon \mathbb{N} \to P$ be the identity. The conclusion of $\mathsf{wpo(set)}$ gives an infinite set $A$ such that $x < y$ implies $x \leq_P y$ for all $x, y \in A$. Therefore $A$ is an infinite $1$-homogeneous set. □

**Corollary 3.5.6.** *Over* $\mathsf{RCA}_0$, $\mathsf{wpo(ext)} \to \mathsf{wpo(set)}$ *implies* $\mathsf{CAC}$.

*Proof.* It follows from Theorem 3.5.5 since $\mathsf{wpo} \to \mathsf{wpo(ext)}$ is provable in $\mathsf{RCA}_0$. □

## Application to well-scattered partial orders

For all $n \geq 1$, we consider the statement:

$\mathsf{st\text{-}ER}^2_{n+1}$: every semitransitive coloring $c\colon [\mathbb{Q}]^2 \to n+1$ has either an infinite $i$-homogeneous set for some $i < n$ or a dense $n$-homogeneous set.

**Question 3.5.7.** Over $\mathsf{RCA}_0$, is $\mathsf{st\text{-}ER}^2_2$ equivalent to $\mathsf{st\text{-}ER}^2_{n+1}$ for all $n \geq 1$?

We establish the following facts about $\mathsf{st\text{-}ER}^2_{n+1}$. In particular, the principle for $n = 2$ (3 colors) is equivalent to that for $n \geq 2$.

**Lemma 3.5.8.** $\mathsf{RCA}_0$ *proves:*

(1) $(\forall n \geq 1)(\mathsf{st\text{-}ER}^2_{n+2} \Rightarrow \mathsf{st\text{-}ER}^2_{n+1})$*;*

(2) $\mathsf{st\text{-}ER}^2_3 \Leftrightarrow \mathsf{st\text{-}ER}^2_2 \wedge \mathsf{st\text{-}RT}^2_2$*;*

(3) $\mathsf{st\text{-}ER}^2_3 \Rightarrow \mathsf{st\text{-}ER}^2_{n+1}$, *for all* $n \geq 1$*;*

(4) $\mathsf{st\text{-}ER}^2_2 \Rightarrow \mathsf{RT}^1_{<\infty}$.

*Proof.* We argue in $\mathsf{RCA}_0$ and we first prove $(1)$. Let $n \geq 1$ and assume $\mathsf{st\text{-}ER}^2_{n+2}$. Let $c\colon [\mathbb{Q}]^2 \to n+1$ be semitransitive and define $d\colon [\mathbb{Q}]^2 \to n+2$ by setting $d(x,y) = c(x,y)$ if $c(x,y) < n$ and $d(x,y) = n+1$ if $c(x,y) = n$. It is immediate to see that $d$ is semitransitive and so we can apply $\mathsf{st\text{-}ER}^2_{n+2}$ to $d$. A dense $n+1$-homogeneous set for $d$ is a dense $n$-homogeneous set for $c$. Suppose we have an infinite $i$-homogeneous set $A$ for some $i < n+1$. Then $i < n$ and $A$ is an infinite $i$-homogeneous set for $c$.

Let us show $(2)$. We first consider the left to right direction. By $(1)$, it is enough to show that $\mathsf{st\text{-}ER}^2_3$ implies $\mathsf{st\text{-}RT}^2_2$ Let $c\colon [\mathbb{N}]^2 \to 2$ be a semitransitive coloring and define $d\colon [\mathbb{Q}]^2 \to 3$ by letting for all $x <_{\mathbb{Q}} y$

$$d(x,y) := \begin{cases} 0 & \text{if } c(x,y) = 0, \\ 1 & \text{if } c(x,y) = 1 \wedge x < y, \\ 2 & \text{if } c(x,y) = 1 \wedge x > y. \end{cases}$$

It is straightforward to see that $d$ is semitransitive. Now, any homogeneous set for $d$, infinite or dense, is an infinite homogeneous set for $c$. We next consider the other direction. Suppose $\mathsf{st\text{-}ER}^2_2$ and $\mathsf{st\text{-}RT}^2_2$ and let $c\colon [\mathbb{Q}]^2 \to 3$ be semitransitive. We thus define

$d\colon [\mathbb{Q}]^2 \to 2$ by setting for all $x <_{\mathbb{Q}} y$

$$d(x,y) := \begin{cases} 0 & \text{if } c(x,y) < 2, \\ 1 & \text{if } c(x,y) = 2. \end{cases}$$

It is easy to see that $d$ is semitransitive as well and so we can apply $\mathsf{st\text{-}ER}^2_2$. If $D$ is a dense 1-homogeneous set for $d$, then $D$ is a dense 2-homogeneous set for $c$ and we are done. Suppose now $A \subseteq \mathbb{Q}$ is an infinite 0-homogeneous set. Therefore $x <_{\mathbb{Q}} y$ implies $c(x,y) < 1$ for all $x, y \in A$. Since $c$ is semitransitive, we can define a partial order on $A$ by letting $x \leq_A y$ if and only if $x = y$ or $x <_{\mathbb{Q}} y$ and $c(x,y) = 1$. By $\mathsf{CAC}$, which is equivalent to $\mathsf{st\text{-}RT}^2_2$, $A$ contains either an infinite antichain, which is an infinite 0-homogeneous set for $c$, or an infinite chain, which is an infinite 1-homogeneous set for $c$.

$(3)$ is proved by induction on $n$. The argument is similar to the right to left direction of $(2)$ by using the fact that $\mathsf{st\text{-}ER}^2_3$ implies $\mathsf{st\text{-}RT}^2_2$.

Finally we show $(4)$. Assume $\mathsf{st\text{-}ER}^2_2$ and let $f\colon \mathbb{N} \to k$. Define a semitransitive coloring $c\colon [\mathbb{Q}]^2 \to 2$ by letting $c(x,y) = 0$ if $f(x) \neq f(y)$ and $c(x,y) = 1$ otherwise. By the finite pigeonhole principle, which is provable in $\mathsf{RCA}_0$, $c$ does not have an infinite antichain. It follows that there is a dense 1-homogeneous set $D \subseteq \mathbb{Q}$. In particular, $D$ is an infinite set and $f(x) = f(y)$ for all $x, y \in D$. Therefore $f^{-1}(i)$ is infinite where $i = f(x)$ for some (any) $x \in D$. $\square$

**Lemma 3.5.9.** *Over* $\mathsf{RCA}_0$, $\mathsf{wspo} \to \mathsf{wspo(set)}$ *implies* $\mathsf{st\text{-}RT}^2_2$ *and* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$ *implies* $\mathsf{st\text{-}ER}^2_2$.

*Proof.* Suppose $\mathsf{wspo} \to \mathsf{wspo(set)}$. Let $c\colon [\mathbb{N}]^2 \to 2$ be semitransitive. Define a partial order $P$ by setting $x \leq_P y$ if and only if $x = y$ or $x < y$ and $c(x,y) = 1$. If $P$ is $\mathsf{wspo(set)}$, let $f\colon \mathbb{Q} \to P$ be the identity. Then there exists an infinite set $A$ such that $x <_{\mathbb{Q}} y$ implies $x \leq_P y$ for all $x, y \in A$. Hence, $c(x,y) = 1$ for all $x, y \in A$ with $x \neq y$ and $A$ is an infinite 1-homogeneous set. Suppose $P$ is not $\mathsf{wspo(set)}$ and so is not $\mathsf{wspo}$. By definition, there is a function $f\colon \mathbb{Q} \to P$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \nleq_P f(y)$ for all $x, y \in \mathbb{Q}$. Provably in $\mathsf{RCA}_0$, we can define an infinite set $A \subseteq \mathbb{Q}$ such that $x < y$ implies $x <_{\mathbb{Q}} y$ and $f(x) < f(y)$ for all $x, y \in A$. It follows that $x <_{\mathbb{Q}} y$ implies $c(f(x), f(y)) = 0$ and so $f(A)$ is an infinite 0-homogeneous set. Notice that $f(A)$ is $\boldsymbol{\Delta}^0_1$ and so exists in $\mathsf{RCA}_0$.

We next assume $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$ and show $\mathsf{st\text{-}ER}^2_2$. Let $c\colon [\mathbb{Q}]^2 \to 2$ be semitransitive. By definition $c$ is transitive on $1$ and hence we can define a partial order $P$ by letting $x \leq_P y$ if and only if $x = y$ or $x <_{\mathbb{Q}} y$ and $c(x, y) = 1$. Consequently $x \leq_P y$ implies $x \leq_{\mathbb{Q}} y$ and so $\mathbb{Q}$ is a linear extension of $P$ showing that $P$ does not satisfy $\mathsf{wspo(ext)}$. Therefore $P$ does not satisfy $\mathsf{wspo(ant)}$. An infinite antichain of $P$ is an infinite $0$-homogeneous set. On the other hand, a dense subchain of $P$ is a dense $1$-homogeneous set. □

**Theorem 3.5.10.** *Over* $\mathsf{RCA}_0$*,*

(1) $\mathsf{st\text{-}ER}^2_3$ *is equivalent to* $\mathsf{wspo(ant)} \to \mathsf{wspo(set)}$*;*

(2) $\mathsf{st\text{-}ER}^2_2$ *is equivalent to* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$.

*Proof.* We argue in $\mathsf{RCA}_0$ and show $(1)$. We first assume $\mathsf{st\text{-}ER}^2_3$ and prove $\mathsf{wspo(ant)} \to \mathsf{wspo(set)}$. Let $P$ be a scattered FAC partial order and $f\colon \mathbb{Q} \to P$. Define a coloring $c\colon [\mathbb{Q}]^2 \to 3$ by letting for all $x <_{\mathbb{Q}} y$

$$c(x, y) := \begin{cases} 0 & \text{if } f(x) \perp_P f(y), \\ 1 & \text{if } f(x) \leq_P f(y), \\ 2 & \text{if } f(x) >_P f(y). \end{cases}$$

It is easy to see that $c$ is transitive on $1$ and $2$ and so is semitransitive. Apply $\mathsf{st\text{-}ER}^2_3$. An infinite $0$-homogeneous set yields an infinite antichain, contradicting $P$ FAC, and a dense $2$-homogeneous set yields an embedding of $\mathbb{Q}^*$ into $P$ contradicting $P$ scattered. Therefore we get an infinite $1$-homogeneous set $A$ which satisfies the conclusion of $\mathsf{wspo(set)}$.

We now consider the other direction. By theorem 3.5.9, since $\mathsf{wspo} \to \mathsf{wspo(ant)}$ and $\mathsf{wspo(set)} \to \mathsf{wspo(ext)}$ are provable in $\mathsf{RCA}_0$, $\mathsf{wspo(ant)} \to \mathsf{wspo(set)}$ implies both $\mathsf{st\text{-}RT}^2_2$ and $\mathsf{st\text{-}ER}^2_2$. By Lemma 3.5.8 $(2)$, $\mathsf{wspo(ant)} \to \mathsf{wspo(set)}$ implies $\mathsf{st\text{-}ER}^2_3$.

Let us consider $(2)$. The right to left direction is proved in Lemma 3.5.9. Assume $\mathsf{st\text{-}ER}^2_2$ and let $P$ be a partial order. We prove the contrapositive of $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$. So let $L$ be a nonscattered linear extension of $P$ and $f\colon \mathbb{Q} \to L$ be an embedding. Let us define a semitransitive coloring $c\colon [\mathbb{Q}]^2 \to 2$ by letting for all $x <_{\mathbb{Q}} y$

$$c(x, y) := \begin{cases} 0 & \text{if } f(x) \perp_P f(y), \\ 1 & \text{if } f(x) <_P f(y). \end{cases}$$

If $A \subseteq \mathbb{Q}$ is an infinite 0-homogeneous set, then $\operatorname{ran}(f)$ is an infinite antichain of $P$. Provably in $\mathsf{RCA}_0$, any $\mathbf{\Sigma}^0_1$ infinite set contains a $\mathbf{\Delta}^0_1$ infinite subset and hence $\operatorname{ran}(f)$ contains an infinite antichain. Suppose we have a dense 1-homogeneous set $D$. Then the restriction of $f$ to $D$ is an embedding of a dense linear order into $P$ showing that $P$ is not scattered. This completes the proof. □

**Corollary 3.5.11.** *Over* $\mathsf{RCA}_0$*,* $\mathsf{wspo}(\mathsf{ant}) \to \mathsf{wspo}(\mathsf{ext})$ *implies* $\mathsf{RT}^1_{<\infty}$ *and hence* $\mathsf{WKL}_0$ *does not prove* $\mathsf{wspo}(\mathsf{ant}) \to \mathsf{wspo}(\mathsf{ext})$.

*Proof.* Recall that $\mathsf{WKL}_0$ does not prove $\mathsf{RT}^1_{<\infty}$ because $\mathsf{WKL}_0$ is $\mathbf{\Pi}^1_1$-conservative over $\mathsf{RCA}_0$ (Harrington, see for instance [Sim09, Theorem IX.2.1]) and $\mathsf{RT}^1_{<\infty}$ is a $\mathbf{\Pi}^1_1$ statement. □

## 3.6 Unprovability

Consider the following result from [CMS04].

**Theorem 3.6.1.** *There exists a computable partial order $P$ such that $P$ has a computable infinite antichain and every computable linear extension is computably well-ordered.*

**Corollary 3.6.2.** $\mathbf{REC}$ *does not satisfy* $\mathsf{wspo}(\mathsf{ext}) \to \mathsf{wspo}(\mathsf{ant})$ *and hence* $\mathsf{RCA}_0$ *does not prove* $\mathsf{wspo}(\mathsf{ext}) \to \mathsf{wspo}(\mathsf{ant})$.

*Proof.* Let $P$ as in Theorem 3.6.1. $P$ clearly does not satisfy $\mathsf{wspo}(\mathsf{ant})$ in $\mathbf{REC}$. On the other hand, every computable linear extension of $P$ is computably well-ordered, and in particular computably scattered. Hence, $P$ satisfies $\mathsf{wspo}(\mathsf{ext})$ in $\mathbf{REC}$. □

We show how to modify the proof of Theorem 3.6.1 to obtain that $\mathsf{WWKL}_0$ does not prove $\mathsf{wpo}(\mathsf{ext}) \to \mathsf{wpo}(\mathsf{ant})$.

**Theorem 3.6.3.** *There exists a computable partial order $P$ with an infinite computable antichain such that the set of reals computing a linear extension $L$ of $P$ and an infinite descending sequence in $L$ is null.*

*Proof.* We want to define a computable partial order $P = (\omega, \preceq)$ such that $\mathcal{A} = \bigcup_{e,i\in\omega} \mathcal{A}_{e,i}$ is null, where $\mathcal{A}_{e,i}$ is the set of $X \in 2^\omega$ such that $\Phi^X_e$ is a linear extension of $P$ and $\Phi^X_i$ is an infinite descending sequence in $\Phi^X_e$. We also ensure the existence of a computable infinite antichain by making $3n \perp 3m$ for all $n \neq m$.

The construction of $P$ is finite injury. We fix a rational $\delta$ with $\frac{4}{5} \leq \delta < 1$ and we meet the requirement $R_{e,i}$: $\mu(\mathcal{A}_{e,i}) \leq \delta$ for all $e, i \in \omega$. This is enough to ensure that $\mathcal{A}$ is null. In fact, suppose $\mu(\mathcal{A}) > 0$ and fix $0 \leq \delta < 1$. Let $e', i'$ be such that $\mu(\mathcal{A}_{e',i'}) > 0$. By the Lebesgue density theorem, there is $\sigma \in 2^{<\omega}$ such that $\mu(\mathcal{A}_{e',i'} \cap [\sigma]) > \delta \cdot 2^{-|\sigma|}$. Now let $e, i$ be such that for all $X$, $\Phi_e^X = \Phi_{e'}^{\sigma^\frown X}$ and $\Phi_i^X = \Phi_{i'}^{\sigma^\frown X}$. Then $\mu(\mathcal{A}_{e,i}) = 2^{|\sigma|}\mu(\mathcal{A}_{e',i'}) > \delta$.

At each stage $s+1$ we add three new points $3s, 3s+1, 3s+2$ and define the restriction of $P$ to $\{0, 1, \ldots, 3s+2\}$. We make the new points pairwise incomparable and place them between two points $u_{s+1}, v_{s+1}$ (the ones taking care of higher priority requirements), where $u_{s+1} \prec v_{s+1}$ if both defined. This means that if $i < 3s$ and $j \in \{3s, 3s+1, 3s+2\}$, then

- $i \preceq j$ if and only if $u_{s+1}$ is defined and $i \preceq u_{s+1}$;
- $j \preceq i$ if and only if $v_{s+1}$ is defined and $v_{s+1} \preceq i$.

We meet a single requirement $R_{e,i}$ by fixing two incomparable points $x$ and $y$ of the form $3n+1$ and $3n+2$ respectively and waiting for a stage $s+1$ such that:

$$\mu(\{X \colon \Phi_e^X(x, y)[s] \downarrow 1\}) \geq \delta/2 \text{ or } \mu(\{X \colon \Phi_e^X(x, y)[s] \downarrow 0\}) \geq \delta/2.$$

If this never happens, then $\mu(\{X \colon \Phi_e^X$ is a linear extension of $P\}) \leq \delta$ and $R_{e,i}$ is satisfied. Otherwise, if at stage $s+1$ we see $\mu(\{X \colon \Phi_e^X(x, y)[s] \downarrow 1\}) \geq \delta/2$ (the other case is similar), then we start building below $x$ waiting for a stage $t+1$ such that

$$\mu(\{X \colon (\exists n < t)\Phi_i^X(n)[t] \downarrow \preceq x\}) \geq \delta.$$

If we never see such a stage, then $(\forall^\infty z)(z \preceq x)$. Therefore,

$$\mu(\{X \colon \Phi_i^X \text{ is a descending sequence in } \Phi_e^X\}) \leq \mu(X \colon (\exists n)\Phi_i^X(n) \downarrow \preceq x\}) \leq \delta$$

and $R_{e,i}$ satisfied. Otherwise, after stage $t+1$ we start building above $y$ for the rest of the construction so that $(\forall^\infty z)(y \preceq z)$. Therefore,

$$\begin{aligned}&\mu(\mathcal{A}_{e,i}) = \mu(\mathcal{A}_{e,i} \cap \{X \colon \Phi_e^X(x, y) \downarrow 1\}) + \mu(\mathcal{A}_{e,i} \cap \{X \colon \Phi_e^X(x, y) \downarrow 0\}) \leq \\ &1 - \mu\{X \colon (\exists n)\Phi_i^X(n) \downarrow \preceq x\}) + \mu(\{X \colon \Phi_e^X(x, y) \downarrow 0\}) \leq (1-\delta) + (1-\delta/2) \leq \delta,\end{aligned}$$

and $R_{e,i}$ is satisfied again.

**Construction.**

Stage $s = 0$. Let $u_0, v_0$ be undefined and $r_0(e, i) = 0$ for all $e, i \in \omega$.

Stage $s+1$. Search for the least $(e, i) < s$ such that $R_{e,i}$ has witnesses $x = 3n+1, y = 3n + 2$ and one of the following holds:

a) $r_s(e, i) = 0$ and either

$$\mu(\{X\colon \Phi_e^X(x, y)[s] \downarrow 1\}) \geq \delta/2 \text{ or } \mu(\{X\colon \Phi_e^X(x, y)[s] \downarrow 0\}) \geq \delta/2;$$

b) $r_s(e, i) = z \in \{x, y\}$ and $\mu(\{X\colon (\exists n < s)\Phi_i^X(n)[s] \downarrow\preceq z\}) \geq \delta$.

If there is no requirement as above, let all the parameters unchanged. Otherwise, $R_{e,i}$ acts.

Suppose a) holds. If $\mu(\{X\colon \Phi_e^X(x, y)[s] \downarrow 1\}) \geq \delta/2$, let $v_{s+1} = r_{s+1}(e, i) = x$, otherwise let $v_{s+1} = r_{s+1}(e, i) = y$. In either case, let $u_{s+1} = u_{n+1}$.

Suppose b) holds. Let $u_{s+1} = y$ if $r_s(e, i) = x$ and $u_{s+1} = x$ otherwise. In either case, let $v_{s+1} = v_{n+1}$ and $r_{s+1}(e, i) = -1$.

Then cancel all witnesses of lower priority requirements, and let $r_{s+1}(e', i') = 0$ for $(e', i') > (e, i)$ and $r_{s+1}(e', i') = r_s(e', i')$ for $(e', i') < (e, i)$. Finally, attach witnesses $3s + 1$ and $3s + 2$ to the least requirement with no witnesses and add $3s, 3s + 1, 3s + 2$ accordingly.

**Claim.** For all $n \neq m$, $3n \perp_P 3m$.

It is quite straightforward to verify by induction that every point marked $u_s$ and $v_s$ is of the form $3n + 1$ or $3n + 2$ and $u_s \neq v_t$ for all $s, t$. Besides, again by induction, it is easy to check that:

- $i < j$ and $i \preceq j$ implies $i = u_s$ for some $s$;
- $i < j$ and $i \succeq j$ implies $i = v_s$ for some $s$.

It follows that $3n \perp j$ for all $n$ and for all $j > 3n$. The claim thus follows.

**Claim.** Every requirement acts finitely often and is satisfied.

The usual inductive argument shows that the strategy for a single requirement succeeds in the full construction. $\square$

**Corollary 3.6.4.** $\mathsf{WWKL}_0$ *does not prove* $\mathsf{wpo}(\mathsf{ext}) \to \mathsf{wpo}(\mathsf{ant})$.

*Proof.* The same argument of Corollary 2.5.4 applies. Let $P$ be as in Theorem 3.6.3 and $X$ be a Martin-Löf random real such that every linear extension of $P$ computed by $X$ has no descending sequences computable in $X$. Now let $M$ be an $\omega$-model of $\mathsf{WWKL}_0$ such that any set in $M$ is computable in $X$. It is clear that in $M$ the partial order $P$ is not FAC and yet every linear extension is well-founded. □

**Corollary 3.6.5.** $\mathsf{WWKL}_0$ *does not prove* $\mathsf{wspo(ext)} \to \mathsf{wspo(ant)}$.

*Proof.* The same $\omega$-model of $\mathsf{WWKL}_0$ works since any dense chain on a partial order computes a descending sequence. □

We finally show that $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$ fails in $\mathbf{REC}$.

**Theorem 3.6.6.** $\mathbf{REC}$ *does not satisfy* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$ *and hence* $\mathsf{RCA}_0$ *does not prove* $\mathsf{wspo(ant)} \to \mathsf{wspo(ext)}$.

*Proof.* Fix a computable presentation of the rationals $\mathbb{Q} = (\omega, \leq_{\mathbb{Q}})$. We build a computable partial order $P = (\omega, \leq_P)$ such that $\mathbb{Q}$ is a linear extension of $P$. Thus, whenever we make two elements $P$-comparable, we do this consistently with $\mathbb{Q}$.

We build $P$ in stages. At stage $s+1$, we define the restriction to $\{0, 1, \dots, s\}$. We make sure that $P$ is scattered and has no infinite antichains in $\mathbf{REC}$ by meeting the following requirements:

$R_{2e}$: if $W_e$ is infinite, then $W_e$ is not an infinite antichain on $P$;

$R_{2e+1}$: if $W_e$ is an infinite chain on $P$, then $W_e$ is not dense.

The priority order is $R_0, R_1, R_2, \dots$ A requirements acts by choosing a *witness*. A witness of $R_{2e}$ is single point $x$, while a witness of $R_{2e+1}$ is a pair of points $(y_0, y_1)$ such that $y_0 <_P y_1$. We call a witness for $R_{2e}$ *positive*. The idea is that we force new points to be comparable with the positive ones: at stage $s+1$ we make $s$ comparable with $z < s$ if and only if there exists a positive point $x < s$ such that either $s <_{\mathbb{Q}} x \leq_P z$ or $z \leq_P x <_{\mathbb{Q}} s$. We guarantee transitivity at each stage. To make $P$ computable, at each stage we ensure that the positive points are pairwise comparable, otherwise we would force incomparable points to become comparable at a later stage.

We satisfy $R_{2e}$ by making sure that if $W_e$ is infinite then there exists $x \in W_e$ such that $x$ is eventually positive. We satisfy $R_{2e+1}$ by making sure that if $W_e$ is an infinite chain then there exist $y_0 <_P y_1$ in $W_e$ such that $(y_0, y_1)_P$ is finite.

We say that a requirement $R_{2e}$ requires attention at stage $s+1$ if $e < s$, $R_{2e}$ has no witness and there exists $x \in W_{e,s}$ such that $x$ is comparable with any positive point of higher priority and for no witness $(y_0, y_1)$ of higher priority $y_0 \leq_P x \leq_P y_1$.

We say that a requirement $R_{2e+1}$ requires attention at stage $s+1$ if $e < s$, $R_{2e+1}$ has no witness and there exist $y_0, y_1 \in W_{e,s}$ such that $y_0 <_P y_1$ and for no positive point $x$ of higher priority $y_0 \leq_P x \leq_P y_1$.

**Construction.**

Stage $s = 0$. Do nothing. In particular, no requirement has a witness.

Stage $s+1$. Search for the highest priority requirement $R_i$ which requires attention. If there is no such requirement, do nothing. Otherwise, such requirement acts by choosing the $\omega$-least witness for which it requires attention. This means that $R_{2e}$ picks the $\omega$-least $x$ and $R_{2e+1}$ picks the $\omega$-least pair $(y_0, y_1)$.

Initialize all lower priority requirements by canceling their witnesses (if any). Add $s$ accordingly.

**Claim.** Every requirements $R_i$ requires attention finitely often and is satisfied.

By induction on $i$. Suppose that every requirement of priority higher than $R_i$ does not require attention after stage $s$ and let $s$ be the least. Then $R_i$ is initialized at stage $s$ and it requires attention at most once after stage $s$. Moreover, witnesses of higher priority requirements never change after stage $s$. So let $A_i$ be the set of positive points $x$ of higher priority and $N_i$ be the set of witnesses $(y_0, y_1)$ of higher priority at the end of stage $s$. Then every point $\geq s$ is comparable with any point of $A_i$. We may assume by induction that there is no $x \in A_i$ such that $y_0 \leq_P x \leq_P y_1$ with $(y_0, y_1) \in N_i$. In particular, $(y_0, y_1)_P$ does not contain points $\geq s$ for every $(y_0, y_1) \in N_i$.

Case $i = 2e$. Suppose $W_e$ is infinite. Then $W_e$ contains a point $\geq s$. In particular, $W_e$ contains a point $x$ which is comparable with any point of $A_i$ and does not belong to $(y_0, y_1)_P$ for all $(y_0, y_1) \in N_i$. Let $t > e, s$ be least such that $W_{e,t}$ contains a point $x$ as above. Then $R_e$ receives attention at stage $t+1$ and picks the $\omega$-least $x \in W_{e,t}$ with the desired property. After stage $t+1$, $R_{2e}$ is never initialized and so every point $> t$ is comparable with $x$.

Case $i = 2e+1$. Suppose $W_e$ is an infinite chain. Since $|A_i| \leq e$ and $W_e$ has $\geq e+2$ comparable points, by the finite pigeonhole principle there must be $y_0, y_1 \in W_e$ such that $y_0 <_P y_1$ and for no $x \in A_i$ we have $y_0 \leq_P x \leq y_1$. Let $t > e, s$ be least such that $W_{e,t}$ contains $y_0, y_1$ as above. Then $R_{2e+1}$ acts at stage $t+1$ by choosing the $\omega$-least pair $(y_0, y_1)$ as required. Every lower priority requirement is initialized at stage $t+1$. Moreover, by induction hypothesis, at stage $t+1$ there are no positive points of higher priority in $(y_0, y_1)_P$. After stage $t+1$, $R_{2e+1}$ is never initialized and so no point $x$ such that $y_0 \leq_P x \leq_P y_1$ is declared positive. Therefore the interval $(y_0, y_1)_P$ does not contain positive points after stage $t+1$ and so no point $> t$ is placed between $y_0$ and $y_1$. Hence $(y_0, y_1)_P$ is finite.

Notice that in either case the induction hypothesis is preserved by construction. $\square$

# 4

# Cardinality of initial intervals[1]

## 4.1 Introduction

Bonnet [Bon75] proves the following result, which is also featured in Fraïssé's monograph [Fra00, §6.7]:

**Theorem 4.1.1.** *If an infinite partial order $P$ is scattered and FAC, then the set of initial intervals of $P$ has the same cardinality of $P$.*

The converse is in general false, but it holds when $|P| < 2^{\aleph_0}$, and in particular when $P$ is countable. Therefore we study the reverse mathematics strength of the following:

**Theorem 4.1.2.** *A countable partial order $P$ is scattered and FAC if and only if the set of initial intervals of $P$ is countable.*

It turns out that the "hard" direction (left to right) of Theorem 4.1.2 is equivalent to $\mathsf{ATR}_0$ (over $\mathsf{ACA}_0$), and the easy one (right to left) is provable in $\mathsf{WKL}_0$ but not in $\mathsf{RCA}_0$. As for Theorem 2.1.1, we are not able to prove the equivalence with $\mathsf{WKL}_0$, and thus we obtain an interesting statement from the point of view of reverse mathematics.

## 4.2 Preliminaries

### 4.2.1 The set of initial intervals

For a partial order $P$, let $\mathcal{I}(P)$ be the set of initial intervals of $P$. In Second Order Arithmetic, $\mathcal{I}(P)$ does not formally exist. Therefore, $I \in \mathcal{I}(P)$ is a shorthand for the formula "$I$ is an initial interval of $P$".

[1]The content of this chapter appears in [FM14]

We say that the partial order $P$ has *countably many initial intervals* if there exists a sequence $\{I_n \colon n \in \mathbb{N}\}$ such that for every $I \in \mathcal{I}(P)$ there exists $n \in \mathbb{N}$ such that $I = I_n$. Otherwise, we say that $P$ has *uncountably many initial intervals*.

Within $\mathsf{ACA}_0$ (but apparently not in weaker systems) we can prove that if $P$ has countably many initial intervals there exists a sequence $\{I_n \colon n \in \mathbb{N}\}$ such that $I \in \mathcal{I}(P)$ if and only if there exists $n \in \mathbb{N}$ such that $I = I_n$. In this case we write $\mathcal{I}(P) = \{I_n \colon n \in \mathbb{N}\}$.

The partial order $P$ has *perfectly many initial intervals* if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$, that is, for all $f \in [T]$, the set $\{x \in \mathbb{N} \colon f(x) = 1\} \in \mathcal{I}(P)$.

A useful tool for studying the notions we just defined is the *tree of finite approximations of initial intervals* of the partial order $P$. We define the tree $T(P) \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in T(P)$ if and only if for all $x, y < |\sigma|$:

- $\sigma(x) = 1$ implies $x \in P$;
- $\sigma(y) = 1$ and $x \preceq y$ imply $\sigma(x) = 1$.

**Lemma 4.2.1** ($\mathsf{RCA}_0$)**.** *Let $P$ be a partial order.*

(1) *$P$ has countably many initial intervals if and only if $T(P)$ has countably many paths;*

(2) *$P$ has perfectly many initial intervals if and only if $T(P)$ contains a perfect subtree.*

*Proof.* Immediate once we notice that the paths in $T(P)$ are exactly the characteristic functions of the initial intervals of $P$. $\square$

In particular, the formula "$P$ has perfectly many initial intervals" is provably $\mathbf{\Sigma}^1_1$ within $\mathsf{RCA}_0$. Moreover, $\mathsf{RCA}_0$ proves that a nonempty perfect tree has uncountably many paths (see Lemma 1.5.3). Therefore we have that $\mathsf{RCA}_0$ proves that a partial order with perfectly many initial intervals has uncountably many initial intervals. Using the perfect tree theorem (see Theorem 1.3.5) we obtain that $\mathsf{ATR}_0$ proves that a partial order with uncountably many initial intervals has actually perfectly many initial intervals. This implies that the formula "$P$ has uncountably many initial intervals" is provably $\mathbf{\Sigma}^1_1$ within $\mathsf{ATR}_0$.

Let us recall the following result due to Peter Clote [Clo89]:

**Theorem 4.2.2** ($\mathsf{ACA}_0$)**.** *The following are equivalent:*

(1) $\mathsf{ATR}_0$*;*

(2) *any countable linear order has countably many or perfectly many initial intervals;*

(3) *any scattered linear order has countably many initial intervals.*

Clote actually states the equivalence of $\mathsf{ATR}_0$ only with (2), but his proofs yield also the equivalence with (3).

### 4.2.2 The system $\mathrm{ATR}_0^X$

Recall that, by [Sim09, Theorem VIII.3.15], $\mathsf{ATR}_0$ is equivalent over $\mathsf{ACA}_0$ to the statement

$$(\forall X)(\forall a \in \mathcal{O}^X)(H_a^X \text{ exists})$$

where $\mathcal{O}^X$ is the collection of (indices for) $X$-computable ordinals and $H_a^X$ codes the iteration of the jump along $a$ starting from $X$. This naturally leads to consider lightface versions of $\mathsf{ATR}_0$, as in [Tan89], [Tan90], and [Mar91]. Here we make explicit mention of the set parameter we use (rather then deal only with the parameterless case and then invoke a relativization process) and let $\mathrm{ATR}_0^X$ be $\mathsf{ACA}_0$ plus the formula $(\forall a \in \mathcal{O}^X)(H_a^X \text{ exists})$. In $\mathrm{ATR}_0^X$ one can prove arithmetical transfinite recursion along any $X$-computable well-order.

By checking the proof of the perfect tree theorem in $\mathsf{ATR}_0$ one readily realizes that $\mathrm{ATR}_0^X$ proves the theorem for $X$-computable trees:

**Theorem 4.2.3** ($\mathrm{ATR}_0^X$)**.** *Every $X$-computable tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with uncountably many paths contains a perfect subtree.*

The following is [Sim09, Lemma VIII.4.19]:

**Theorem 4.2.4** ($\mathrm{ATR}_0^X$)**.** *There exists a countable coded $\omega$-model $M$ such that $X \in M$ and $M$ satisfies $\mathbf{\Sigma}_1^1$-$\mathsf{DC}_0$.*

We will use the following corollary:

**Corollary 4.2.5** ($\mathsf{ATR}_0$)**.** *For all $X$ and $Y$ there exists a countable coded $\omega$-model $M$ such that $X, Y \in M$, and $M$ satisfies both $\mathbf{\Sigma}_1^1$-$\mathsf{DC}_0$ and $\mathrm{ATR}_0^X$.*

*Proof.* We argue in $\mathsf{ATR}_0$ and let $X$ and $Y$ be given. By $\mathbf{\Sigma}_1^1$-$\mathsf{AC}_0$, which is a consequence of $\mathsf{ATR}_0$, the main axiom of $\mathrm{ATR}_0^X$ is equivalent to a $\mathbf{\Sigma}_1^1$ formula $(\exists Z)\varphi(Z, X)$ with $\varphi$ arithmetic. This formula is true in $\mathsf{ATR}_0$, and hence we can fix $Z$ such that $\varphi(Z, X)$. By

Theorem 4.2.4 there exists a countable coded $\omega$-model $M$ of $\mathbf{\Sigma}^1_1$-$\mathsf{DC}_0$ such that $X \oplus Y \oplus Z \in M$. In particular, $X, Y \in M$ and, as $Z \in M$ and $M$ is a model of $\mathbf{\Sigma}^1_1$-$\mathsf{DC}_0$ (hence also of $\mathbf{\Sigma}^1_1$-$\mathsf{AC}_0$), $M$ satisfies $\mathrm{ATR}^X_0$. $\square$

## 4.3 Equivalences with $\mathsf{ATR}_0$

We consider the left to right direction of Theorem 4.1.2, i.e. the statement every countable scattered FAC partial order has countably many initial intervals. We start with a technical Lemma:

**Lemma 4.3.1** ($\mathsf{ACA}_0$). *If a partial order $P$ has perfectly many initial intervals, then there exists $x \in P$ such that either*

(1) $P(\perp x)$ *has uncountably many initial intervals, or*

(2) *both $P(\prec x)$ and $P(\succ x)$ have uncountably many initial intervals.*

*Proof.* Let $P$ be a partial order with perfectly many initial intervals. Let $T \subseteq T(P)$ be a perfect tree.

We first show that there exist $x \in P$ such that both

$$\{I \in \mathcal{I}(P) \colon x \notin I\} \text{ and } \{I \in \mathcal{I}(P) \colon x \in I\}$$

are uncountable. Let $\tau \in T$ be such that both $\tau_0 = \tau^\frown\langle 0\rangle$ and $\tau_1 = \tau^\frown\langle 1\rangle$ belong to $T$. Let $x = |\tau|$ and notice that $x \in P$. For $i < 2$ define $T_i = \{\sigma \in T \colon \sigma \subseteq \tau_i \vee \tau_i \subseteq \sigma\}$. The trees $T_0$ and $T_1$ are perfect and witness the fact that the two collections of initial intervals are uncountable.

Now, suppose that condition (1) fails and let $\mathcal{I}(P(\perp x)) = \{J_n \colon n \in \mathbb{N}\}$. We aim to show that (2) holds.

Suppose for a contradiction that $P(\prec x)$ has countably many initial intervals and let $\mathcal{I}(P(\prec x)) = \{I_n \colon n \in \mathbb{N}\}$. Then it is not difficult to show that

$$\{I \in \mathcal{I}(P) \colon x \notin I\} = \{I_n \cup {\downarrow} J_m \colon n, m \in \mathbb{N}\}.$$

This contradicts the fact that $\{I \in \mathcal{I}(P) \colon x \notin I\}$ is uncountable.

Similarly, suppose that $P(\succ x)$ has countably many initial intervals and let $\mathcal{I}(P(\succ x)) = \{I_n \colon n \in \mathbb{N}\}$. Then, it is not difficult to show that

$$\{I \in \mathcal{I}(P) \colon x \in I\} = \{\downarrow(\{x\} \cup I_n \cup J_m) \colon n, m \in \mathbb{N}\}.$$

This contradicts the fact that $\{I \in \mathcal{I}(P) \colon x \in I\}$ is uncountable. Therefore, condition (2) holds. $\square$

**Theorem 4.3.2** ($\mathsf{ATR}_0$)**.** *If a countable partial order $P$ is scattered and FAC, then $P$ has countably many initial intervals.*

*Proof.* Let $P$ be a countable partial order with uncountably many initial intervals.

Let $\mathrm{Fin}(P)$ the set of (codes for) finite subsets of $P$. For all $F, G, H \in \mathrm{Fin}(P)$, let

$$P_{F,G,H} = \bigcap_{x \in F} P(\prec x) \cap \bigcap_{x \in G} P(\succ x) \cap \bigcap_{x \in H} P(\perp x).$$

We want to define a pruned tree $T \subseteq 3^{<\mathbb{N}}$ and a function $f \colon T \to \mathrm{Fin}(P)^3$ such that the following hold (we write $f(\sigma) = (F_\sigma, G_\sigma, H_\sigma)$ and $P_\sigma = P_{f(\sigma)}$):

(i) $f(\langle\rangle) = (\emptyset, \emptyset, \emptyset)$;

(ii) for all $\sigma \in T$, $\sigma^\frown\langle 0\rangle \in T$ if and only if $\sigma^\frown\langle 1\rangle \in T$ if and only if $\sigma^\frown\langle 2\rangle \notin T$ (in other words there are two possibilities: either exactly $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ belong to $T$, or only $\sigma^\frown\langle 2\rangle \in T$);

(iii) if $\sigma^\frown\langle 0\rangle \in T$, then $f(\sigma^\frown\langle 0\rangle) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma)$ and $f(\sigma^\frown\langle 1\rangle) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma)$ for some $x \in P_\sigma$;

(iv) if $\sigma^\frown\langle 2\rangle \in T$, then $f(\sigma^\frown\langle 2\rangle) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$ for some $x \in P_\sigma$.

We first show that if there exist $T$ and $f$ as above, then $P$ is not scattered or it contains an infinite antichain.

First suppose there exists a path $g \in [T]$ such that $g(n) = 2$ for infinitely many $n$. Then let

$$A = \bigcup_{n \in \mathbb{N}} H_{g \restriction n}.$$

It is easy to check, using (iv) and the definition of $P_{F,G,H}$, that $A$ is an infinite antichain.

If there are no paths $g \in [T]$ such that $g(n) = 2$ for infinitely many $n$ then it is easy to see, using (ii), that $T$ is perfect. For all $\sigma^\frown\langle 0\rangle \in T$, let $x_\sigma$ be the unique element of $F_{\sigma^\frown\langle 0\rangle} \setminus F_\sigma$. We claim that

$$D = \{x_\sigma \colon \sigma^\frown\langle 0\rangle \in T\}$$

is a dense subchain of $P$.

We first note that $x_\sigma \neq x_\tau$ for $\sigma, \tau \in T$ with $\sigma \neq \tau$. Now fix distinct $x_\sigma, x_\tau \in D$ with the goal of showing that they are comparable in $P$ and that there exists an element of $D$ strictly between them. First assume that $\sigma$ and $\tau$ are comparable as sequences, let us say $\sigma \subset \tau$. Then, using (iii), $x_\tau \preceq x_\sigma$ if $\sigma^\frown\langle 0\rangle \subseteq \tau$ and $x_\sigma \preceq x_\tau$ if $\sigma^\frown\langle 1\rangle \subseteq \tau$. Suppose $x_\tau \preceq x_\sigma$ (the other case is similar) and let $\eta \in T$ so that $\tau^\frown\langle 1\rangle \subseteq \eta$ and $x_\eta \in D$. Then $x_\tau \prec x_\eta \prec x_\sigma$ by (iii). Suppose now that $\sigma$ and $\tau$ are not one initial segment of the other. We may assume that $\eta^\frown\langle 0\rangle \subseteq \sigma$ and $\eta^\frown\langle 1\rangle \subseteq \tau$, where $\eta$ is the longest common initial segment of $\sigma$ and $\tau$. Then $x_\eta \in D$ and, using (iii) again, $x_\sigma \prec x_\eta \prec x_\tau$.

It remains to show that we can define $T$ and $f$ satisfying (i)–(iv).

By Theorem 4.2.2, $P$ has perfectly many initial intervals. Let $U$ be a perfect subtree of $T(P)$. By Corollary 4.2.5, there exists an $\omega$-model $M$ of $\mathbf{\Sigma}^1_1$-$\mathsf{DC}_0$ such that $P, U \in M$ and $M$ satisfies $\mathrm{ATR}_0^P$.

We recursively define $T$ and $f$ by using $M$ as a parameter. Let $\langle\rangle \in T$ and $f(\langle\rangle) = (\emptyset, \emptyset, \emptyset)$ as required by (i). Note that $M$ satisfies "$T(P_{\langle\rangle})$ contains a perfect subtree". Let $\sigma \in T$ and assume by arithmetical induction that $M$ satisfies "$T(P_\sigma)$ contains a perfect subtree". Since $M$ is a model of $\mathsf{ACA}_0$, by Lemma 4.3.1 applied to $P_\sigma$, there exists $x \in P_\sigma$ such that either

(a) $M$ satisfies "$T(P_\sigma \cap x^\perp)$ has uncountably many paths", or

(b) $M$ satisfies "both $T(P_\sigma \cap P(\preceq x))$ and $T(P_\sigma \cap P(\succeq x))$ have uncountably many paths".

Search the least $x$ with this arithmetical property. If (a) holds (and we can check this arithmetically outside $M$), use $\mathrm{ATR}_0^P$ within $M$ to apply Theorem 4.2.3 to the $P$-computable tree $T(P_\sigma \cap x^\perp)$. We obtain that $M$ satisfies "$T(P_\sigma \cap x^\perp)$ contains a perfect subtree". Thus, let $\sigma^\frown\langle 2\rangle \in T$ and set $f(\sigma^\frown\langle 2\rangle) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$. If (b) holds, then arguing analogously we obtain that $M$ satisfies "both $T(P_\sigma \cap P(\preceq x))$ and $T(P_\sigma \cap P(\succeq x))$ contain perfects subtrees". Thus let $\sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in T$ and set

$$f(\sigma^\frown\langle 0\rangle) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma) \text{ and } f(\sigma^\frown\langle 1\rangle) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma).$$

In any case, (ii)-(iv) are satisfied and the induction hypothesis that $M$ satisfies "$T(P_\sigma)$ contains a perfect subtree" is preserved. $\square$

**Theorem 4.3.3.** *Over* $\mathsf{ACA}_0$*, the following are equivalent:*

(1) $\mathsf{ATR}_0$*;*

(2) *every countable scattered partial order with no infinite antichains has countably many initial intervals;*

(3) *every countable scattered linear order has countably many initial intervals.*

*Proof.* Assume $\mathsf{ACA}_0$. We wish to prove $\mathsf{ATR}_0$. By [Sim09, Theorem V.5.2], $\mathsf{ATR}_0$ is equivalent (over $\mathsf{RCA}_0$) to the statement asserting that for every sequence of trees $\{T_i \colon i \in \mathbb{N}\}$ such that every $T_i$ has at most one path, there exists the set $\{i \in \mathbb{N} \colon [T_i] \neq \emptyset\}$. So let $\{T_i \colon i \in \mathbb{N}\}$ be such a sequence. Let us order each $T_i$ with the Kleene-Brouwer order $\leq_{\mathrm{KB}}$ and define the linear order $L = \sum_{i \in \mathbb{N}} T_i$

We aim to show that $L$ is scattered. By Lemma 1.5.2, it suffices to prove that every $T_i$ is scattered. To this end, we show that if a tree $T$ has at most one path then the Kleene-Brouwer order on $T$ is of the form

$$X + \sum_{n \in \omega^*} Y_n, \qquad (*)$$

where $X$ and the $Y_n$ are (possibly empty) well-orders. Applying Lemma 1.5.2 again, we obtain that $T$ is scattered.

If $T$ has no path, then $\mathsf{ACA}_0$ proves that $\leq_{\mathrm{KB}}$ well-orders $T$, and hence we can take $X = T$ and the $Y_n$'s empty. Now let $f$ be the unique path of $T$. Let $X = \{\sigma \in T \colon (\forall n) \sigma \leq_{\mathrm{KB}} f \restriction n\}$ and $Y_n = \{\sigma \in T \colon f \restriction n+1 \leq_{\mathrm{KB}} \sigma \leq_{\mathrm{KB}} f \restriction n\}$, for all $n \in \mathbb{N}$. It is straightforward to see that $(*)$ holds. We now claim that $X$ is a well-order. Suppose not, and let $(\sigma_n)_{n \in \mathbb{N}}$ be an infinite descending sequence in $X$. Form the tree $T_0 = \{\sigma \in T \colon (\exists n) \sigma \subseteq \sigma_n\}$. Then $T_0$ is not well-founded and so it has a path. As $T_0$ is a subtree of $T$, this path must be $f$. Let $i \in \mathbb{N}$ be such that $\sigma_0 \restriction i = f \restriction i$ and $\sigma_0(i) < f(i)$ (such an $i$ exists because $\sigma_0 \in X$). On the other hand, $f \restriction i+1 \in T_0$, and thus $f \restriction i+1 \subseteq \sigma_n$ for some $n \in \mathbb{N}$. It follows that $\sigma_0 \leq_{\mathrm{KB}} \sigma_n$, a contradiction. To show that each $Y_n$ is a well-order notice that $Y_n = \{\sigma \in T \colon f \restriction n \subset \sigma \wedge f(n) < \sigma(n)\} \cup \{f \restriction n\}$.

Apply (3) to $L$ and let $\mathcal{I}(L) = \{I_n \colon n \in \mathbb{N}\}$. It is easy to check that $T_i$ has a path if and only if

$$(\exists n)\big(\bigcup_{j<i} T_j \subseteq I_n \wedge T_i \nsubseteq I_n \wedge L \setminus I_n \text{ has no least element}\big).$$

Therefore, the set $\{i \in \mathbb{N} \colon [T_i] \neq \emptyset\}$ exists by arithmetical comprehension. $\square$

It is worth noticing that a natural weakening of condition (3) of Theorem 4.3.3 is provable in $\mathsf{RCA}_0$:

**Lemma 4.3.4** ($\mathsf{RCA}_0$). *Every linear order with perfectly many initial intervals is not scattered.*

*Proof.* Let $L$ be a linear order and $T \subseteq T(L)$ be a perfect tree. Define

$$D = \{x \in L \colon (\exists \sigma \in T)(|\sigma| = x \wedge \sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in T)\}.$$

The argument showing that $Q$ is a dense subchain of $L$ is similar to the one in the proof of Theorem 4.3.2. $\square$

## 4.4 A proof in $\mathsf{WKL}_0$ and unprovability results

The next goal is to show that $\mathsf{WKL}_0$ suffices to prove the right to left direction of Theorem 4.1.2, which states that every partial order with countably many initial intervals is scattered and FAC. Indeed, $\mathsf{RCA}_0$ proves the first half of the right to left direction:

**Theorem 4.4.1** ($\mathsf{RCA}_0$). *Every partial order with countably many initial intervals is scattered.*

*Proof.* We show that if $P$ is not scattered, then $P$ has perfectly many initial intervals. By Lemma 3.3.2 we may assume that $P$ contains a dense linear order $D$.

We define by recursion an embedding $f \colon 2^{<\mathbb{N}} \to T(P)$. Thus $T_0 = \{\tau \in T(P) \colon (\exists \sigma \in 2^{<\mathbb{N}})\tau \subseteq f(\sigma)\}$ is a perfect subtree of $T(P)$. Since $\tau \in T_0$ if and only if $(\exists \sigma \in 2^{<\mathbb{N}})(|\sigma| = |\tau| \wedge \tau \subseteq f(\sigma))$, $T_0$ exists in $\mathsf{RCA}_0$.

We say that $x \in P$ is *free for* $\tau \in T(P)$ if

$$(\forall y < |\tau|)((\tau(y) = 1 \implies x \npreceq y) \wedge (\tau(y) = 0 \implies y \npreceq x)).$$

In other words, $x$ is free for $\tau$ if and only if there exist $\tau_0, \tau_1 \in T(P)$ with $\tau \subset \tau_i$ and $\tau_i(x) = i$. Since $T(P)$ is a pruned tree this means that there exist two initial intervals of $P$ whose characteristic function extends $\tau$, one containing $x$ and the other avoiding $x$.

Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(\sigma) = \tau$. Assume by $\mathbf{\Sigma}^0_1$ induction that $D$ contains at least two (and hence infinitely many) elements that are free for $\tau$. Then search for $u \prec x \prec v$ in $D$ that are free for $\tau$. We will define $\tau_0, \tau_1 \in T(P)$ which are extensions of $\tau$ with $|\tau_i| = x + 1$ and $\tau_i(x) = i$. Thus $\tau_0$ and $\tau_1$ are incompatible and we can let $f(\sigma^\frown\langle i\rangle) = \tau_i$.

We show how to define $\tau_0$ (to define $\tau_1$ replace $u$ with $x$ and $x$ with $v$). Since $\{y \in P \colon y < x\}$ is finite, we can find $u', v' \in D$ with $u \prec u' \prec v' \prec x$ such that $u', v' > x$, and for no $y \in P$ with $y < x$ we have $u' \prec y \prec v'$. Given $y < |\tau_0|$ we need to define $\tau_0(y)$, and we proceed by cases (notice that the first three conditions are determined by the fact that we want $\tau_0 \in T(P)$ and $\tau_0 \supseteq \tau$):

- if $y \notin P$ let $\tau_0(y) = 0$;
- if $y \in P$ is not free for $\tau$ because there exists $z < |\tau|$ such that $\tau(z) = 0$ and $z \preceq y$ let $\tau_0(y) = 0$;
- if $y \in P$ is not free for $\tau$ because there exists $z < |\tau|$ such that $\tau(z) = 1$ and $y \preceq z$ let $\tau_0(y) = 1$;
- if $z$ is free for $\tau$ we define $\tau_0(z)$ according to the following cases:
  - (i) if $z \prec u'$, let $\tau_0(z) = 1$;
  - (ii) if $z \succ v'$, let $\tau_0(z) = 0$;
  - (iii) otherwise, let $\tau_0(z) = 0$.

It is not difficult to check that $\tau_0$ extends $\tau$, $\tau_0(x) = 0$ and both $u'$ and $v'$ are free for $\tau_0$, preserving the induction hypothesis. □

With regard to the other half, $\mathsf{RCA}_0$ proves the following.

**Lemma 4.4.2** ($\mathsf{RCA}_0$)**.** *An infinite antichain has perfectly many initial intervals.*

*Proof.* If $P$ is an antichain then the tree $T(P)$ consists of all $\sigma \in 2^{<\mathbb{N}}$ such that $x \notin P$ implies $\sigma(x) = 0$. If $P$ is infinite it is immediate that this tree is perfect and thus Lemma 4.2.1 implies that $P$ has perfectly many initial intervals. □

We now show that $\mathsf{WKL}_0$ suffices to prove the half of the right to left direction which is not provable in $\mathsf{RCA}_0$ (see Theorem 4.4.1). In other words, we study the statement that every partial order with countably many initial intervals is FAC. To do this, we first consider the relation between initial intervals of partial orders contained one into the other.

**Lemma 4.4.3.** *Over* $\mathsf{RCA}_0$, *the following are equivalent:*

(1) $\mathsf{WKL}_0$;

(2) *Let $Q$ and $P$ be partial orders and $f$ be an embedding of $Q$ into $P$. Then*

$$\mathcal{I}(Q) = \{f^{-1}(J) \colon J \in \mathcal{I}(P)\};$$

(3) *Let $Q$ be a subset of a partial order $P$. Then $\mathcal{I}(Q) = \{J \cap Q \colon J \in \mathcal{I}(P)\}$.*

*Proof.* We start with $(1) \Rightarrow (2)$. Let $f \colon Q \to P$ be an embedding. It is easy to check that if $J \in \mathcal{I}(P)$ then $f^{-1}(J) \in \mathcal{I}(Q)$, so that the right to left inclusion is established even in $\mathsf{RCA}_0$.

For the other inclusion fix $I \in \mathcal{I}(Q)$. Let $\varphi(x)$ and $\psi(x)$ be the $\mathbf{\Sigma}^0_1$ formulas $(\exists y \in Q)(y \in I \land x = f(y))$ and $(\exists y \in Q)(y \notin I \land x = f(y))$ respectively. Since $f$ is an embedding and $I$ is an initial interval, we have

$$(\forall x, y \in P)(\varphi(x) \land \psi(y) \implies y \not\leq_P x).$$

Apply $\mathbf{\Sigma}^0_1$ initial interval separation (Lemma 2.2.4) to get $J \in \mathcal{I}(P)$ such that $f(I) \subseteq J$ and $J \cap f(Q \setminus I) = \emptyset$. It is immediate to see that $I = f^{-1}(J)$.

Since the implication $(2) \Rightarrow (3)$ is obvious, it remains to show $(3) \Rightarrow (1)$. Instead of $\mathsf{WKL}_0$, we prove statement (3) of Lemma 2.2.4, i.e. initial interval separation. Let $P$ be a partial order and $A, B \subseteq P$ such that $(\forall x \in A)(\forall y \in B)(y \not\leq_P x)$. Let $Q = A \cup B \subseteq P$ and notice that $A \in \mathcal{I}(Q)$. By (3) there exists $J \in \mathcal{I}(P)$ such that $A = J \cap Q$. It is easy to see that $A \subseteq J$ and $J \cap B = \emptyset$, completing the proof. □

Notice that the obvious proof of the nontrivial direction of (2), namely given $I \in \mathcal{I}(Q)$ let $J$ be the downward closure of $f(I)$, uses arithmetical comprehension.

**Corollary 4.4.4** ($\mathsf{WKL}_0$). *Let $P$ and $Q$ be partial orders such that $Q$ embeds into $P$. If $P$ has countably many initial intervals, then $Q$ does.*

*Proof.* Fix an embedding $f\colon Q \to P$. Let $\{J_n\colon n \in \mathbb{N}\}$ be such that for all $J \in \mathcal{I}(P)$ there exists $n$ with $J = J_n$. For every $n$ let $I_n = f^{-1}(J_n)$ (this can be done in $\mathsf{RCA}_0$). Then for all $I \in \mathcal{I}(Q)$ there exists $n$ with $I = I_n$, showing that $Q$ has countably many initial intervals. □

We can now prove in $\mathsf{WKL}_0$ the part of the right to left direction of Theorem 4.1.2 we are interested in.

**Theorem 4.4.5** ($\mathsf{WKL}_0$)**.** *Every partial order with countably many initial intervals has no infinite antichains.*

*Proof.* Immediate from Lemma 4.4.2 and Corollary 4.4.4. □

Finally, we show that the the right to left direction of Theorem 4.1.2 is not provable in $\mathsf{RCA}_0$. The proof uses the same computable partial order of Lemma 2.5.1.

**Theorem 4.4.6.** $\mathsf{RCA}_0$ *does not prove that every partial order with countably many initial intervals is FAC.*

*Proof.* We show that the statement fails in $\mathbf{REC}$. Let $P$ be the computable partial order of Lemma 2.5.1. Recall that $P$ contains an infinite computable antichain and all computable initial intervals of $P$ are downward closures of finite subsets of $P$.

Since the downward closures of finite subsets of $P$ are uniformly computable, there exists a set $\{I_n\colon n \in N\}$ in $\mathbf{REC}$ which lists all computable initial intervals of $P$. Therefore $\mathbf{REC}$ satisfies that $P$ has countably many initial intervals. Since $P$ has an infinite antichain in $\mathbf{REC}$, the conclusion follows. □

**Note.** Theorem 2.5.3 implies that $\mathsf{WWKL}_0$ does not prove that every partial order with countably many initial intervals is FAC. In fact, by the same argument of Corollary 2.5.4, there exists a computable partial order $P$ and an $\omega$-model $M$ of $\mathsf{WWKL}_0$ such that $P$ has a computable infinite antichain (and hence is not FAC in M) and every initial interval of $P$ which belongs to $M$ is the downward closure of a finite set (and hence $P$ has countably many initial intervals in $M$).

# 5

# Hausdorff's analysis of scattered linear orders

## 5.1 Introduction

In [Hau08], Hausdorff proved the following theorem.

**Theorem 5.1.1.** *The class of scattered linear orders is the least class which contains the empty set, singletons and is closed under sums along $\mathbb{Z}$.*

Along with the above classification theorem, Hausdorff proved that a linear order $L$ is scattered if and only if $\mathrm{rk}_{\mathrm{H}}(L)$ exists. Here, $\mathrm{rk}_{\mathrm{H}}(L)$ denotes the Hausdorff rank of $L$. He also proved that, for every ordinal $\alpha$, $\mathrm{rk}_H(L) \leq \alpha$ if and only if $L$ is embeddable into $\mathbb{Z}^\alpha$, where $\mathbb{Z}^\alpha$ generalizes ordinal exponentiation. For a general discussion on Hausdorff rank and powers of $\mathbb{Z}$ see [Ros82, chapter 5, §4].

In the context of reverse mathematics, Clote [Clo89] proved the following:

**Theorem 5.1.2** ($\mathsf{ATR}_0$)**.** *If $L$ is a scattered linear order, then $\mathrm{rk}_{\mathrm{H}}(L) \leq \alpha$ for some well-order $\alpha$.*

**Theorem 5.1.3** ($\mathsf{ATR}_0$)**.** *Every scattered linear order embeds into $\mathbb{Z}^\alpha$ for some well-order $\alpha$.*

In section 5.3, we supply the details of the proof of Theorem 5.1.3 and show its equivalence with $\mathsf{ATR}_0$. In section 5.4, we show that $\mathsf{ATR}_0$ proves the following theorem featured in [Fra00, §5.3.2].

**Theorem 5.1.4.** *If $L$ is a countable scattered linear order, then there exists a countable ordinal which does not embed into $L$*

Notice that the theorem is not true if $L$ is uncountable: for instance $\omega_1$ is an uncountable scattered linear order but any countable ordinal embeds into it.

Finally, in section 5.5, we prove that Hausdorff's classification theorem (Theorem 5.1.1) is equivalent to $\mathsf{ATR}_0$.

## 5.2 Preliminaries

Recall from [Hir05] the definition of ordinal exponentiation in $\mathsf{RCA}_0$.

**Definition 5.2.1** ($\mathsf{RCA}_0$)**.** Let $\alpha$ be a well-order. We define $\omega^\alpha$ to be the set

$$\{\delta\colon \alpha \to \omega \mid \delta(\beta) = 0 \text{ for all but finitely many } \beta < \alpha\},$$

linearly ordered by $\delta \leq \lambda$ if and only if $\delta = \lambda$ or $\delta(\beta) < \lambda(\beta)$ for the largest $\beta < \alpha$ such that $\delta(\beta) \neq \lambda(\beta)$.

Formally, an element of $\omega^\alpha$ is a sequence $\langle(\beta_0, n_0), \ldots (\beta_k, n_k)\rangle$, where $\beta_{i+1} < \beta_i < \alpha$ and $n_i \in \mathbb{N} \setminus \{0\}$. We usually denote $\langle(\beta_0, n_0), \ldots (\beta_k, n_k)\rangle$ by $\omega^{\beta_0} n_0 + \ldots + \omega^{\beta_k} n_k$.

The empty sequence corresponds to the the constant function $\delta(\beta) = 0$ for all $\beta < \alpha$ and is denoted by $0$.

We will use the following known fact later.

**Theorem 5.2.2** ([Hir05])**.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{ACA}_0$*;*

(2) *If $\alpha$ and $\beta$ are well-orders, then so is $\alpha^\beta$;*

(3) *If $\alpha$ is a well-order, then so is $2^\alpha$.*

Along the same lines (see also [Ros82]), we define $\mathbb{Z}^\alpha$.

**Definition 5.2.3** ($\mathsf{RCA}_0$)**.** Let $\alpha$ be a well-order. We define $\mathbb{Z}^\alpha$ to be the set

$$\{x\colon \alpha \to \mathbb{Z} \mid x(\beta) = 0 \text{ for all but finitely many } \beta\},$$

linearly ordered by $x \leq y$ if and only if $x = y$ or $x(\beta) <_{\mathbb{Z}} y(\beta)$ for the largest $\beta < \alpha$ such that $x(\beta) \neq y(\beta)$.

We write $x <_\beta y$ for $x(\beta) <_{\mathbb{Z}} y(\beta)$ and $x(\gamma) = y(\gamma)$ for all $\beta < \gamma < \alpha$. For all $x \in \mathbb{Z}^\alpha$ and $\beta < \alpha$, let

$$\mathbb{Z}(x)^\beta = \{y \in \mathbb{Z}^\alpha \colon y(\gamma) = x(\gamma) \text{ for all } \gamma \text{ such that } \beta \leq \gamma < \alpha\}.$$

If $x = 0$, we write $\mathbb{Z}^\beta$ instead of $\mathbb{Z}(0)^\beta$. We also let $\mathbb{Z}(x)^\alpha = \mathbb{Z}^\alpha$.

**Definition 5.2.4** ($\mathsf{RCA}_0$)**.** Let $\alpha$ be a well-order. The *ordinal sum* between elements of $\omega^\alpha$ is defined by the rules:

- $\omega^\beta n + \omega^\gamma m = \omega^\gamma m$ for $\beta < \gamma$;
- $\omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$.

We also let $\delta + \omega^\alpha = \omega^\alpha$ for $\delta < \alpha$.

So, for instance, $(\omega^7 5 + \omega^4 2 + \omega^3 2 + 4) + (\omega^4 5 + \omega^2 9) = \omega^7 5 + \omega^4 7 + \omega^2 9$.

**Lemma 5.2.5** ($\mathsf{RCA}_0$)**.** *Let $\alpha$ be a well-order. Then for all $\beta < \alpha$ and $\delta < \omega^\alpha$ there is an isomorphism between $\{\lambda < \omega^\alpha \colon \lambda < \omega^\beta\}$ and $[\delta, \delta + \omega^\beta)$.*

*Proof.* Straightforward. □

We point out the following useful fact. Remember that in $\mathsf{RCA}_0$ we have $\mathbf{\Pi}^0_1$ induction.

**Lemma 5.2.6.** $\mathsf{RCA}_0$ *proves $\mathbf{\Pi}^0_1$ transfinite induction, i.e. for every $\mathbf{\Pi}^0_1$ formula $\varphi(n)$* $\mathsf{RCA}_0$ *proves that for any well-order $\alpha$*

$$(\forall \beta < \alpha)\big((\forall \gamma < \beta)\varphi(\gamma) \implies \varphi(\beta)\big) \implies (\forall \beta < \alpha)\varphi(\beta).$$

*Proof.* Given a well-order $\alpha$, suppose $(\forall \beta < \alpha)\big((\forall \gamma < \beta)\varphi(\gamma) \implies \varphi(\beta)\big)$ and $\neg\varphi(\beta_0)$ for some $\beta_0 < \alpha$. Then define by recursion an infinite descending sequence $(\beta_n)$ in $\alpha$ by letting $\beta_{n+1}$ be the $\omega$-least $\beta < \beta_n$ such that $\neg\varphi(\beta)$. Use $\mathbf{\Sigma}^0_1$ induction to ensure $\neg\varphi(\beta_n)$ for all $n$. □

**Lemma 5.2.7** ($\mathsf{RCA}_0$)**.** *For every well-order $\alpha$, $\mathbb{Z}^\alpha$ is scattered.*

*Proof.* Let $\alpha$ be a well-order. Aiming for a contradiction, let $f \colon \mathbb{Q} \to \mathbb{Z}^\alpha$ be an embedding. Let $\beta < \alpha$ be the least such that

$$(\exists i, j \in \mathbb{Q})(i <_{\mathbb{Q}} j \wedge f(i) <_\beta f(j)).$$

By Lemma 5.2.6, every $\mathbf{\Sigma}^0_1$ subset of a well-order has a least element and so $\beta$ exists. Then seek for $i, j \in \mathbb{Q}$ so that $f(i) <_\beta f(j)$ and $|(f(i)(\beta), f(j)(\beta))_\mathbb{Z}| = n \in \mathbb{N}$ is $\omega$-least. By $\mathbf{\Pi}^0_1$ induction, such $i, j$ exist.

Let $k \in \mathbb{Q}$ with $i <_\mathbb{Q} k <_\mathbb{Q} j$ and set $x = f(i)$, $y = f(j)$ and $z = f(k)$. It follows that $x(\gamma) = z(\gamma) = y(\gamma)$ for all $\beta < \gamma < \alpha$ and $x(\beta) \leq_\mathbb{Z} z(\beta) \leq_\mathbb{Z} y(\beta)$. By the minimality of $n$, either $x(\beta) = z(\beta)$ or $z(\beta) = y(\beta)$. Suppose $x(\beta) = z(\beta)$. Then $x <_\gamma z$ for some $\gamma < \beta$, contrary to the minimality of $\beta$. Similarly for $z(\beta) = y(\beta)$. Each case leads to a contradiction. □

**Lemma 5.2.8** (RCA$_0$)**.** *For every countable well-order $\alpha$, $\mathbb{Z}^\alpha$ has countably many initial intervals.*

*Proof.* Let $\alpha$ be given and let $L = \mathbb{Z}^\alpha$. It is not difficult to see that, for all $x \in L$ and $\beta < \alpha$, $\downarrow\mathbb{Z}(x)^\beta = \mathbb{Z}(x)^\beta \cup L(\preceq x)$. Then every $\downarrow\mathbb{Z}(x)^\beta$ is $\mathbf{\Sigma}^0_0$ definable and thus exists in RCA$_0$. We thus set out to prove that

$$\mathcal{I}(\mathbb{Z}^\alpha) = \{\emptyset\} \cup \{\downarrow\mathbb{Z}(x)^\beta \colon x \in \mathbb{Z}^\alpha \wedge \beta < \alpha\} \cup \{\mathbb{Z}^\alpha\}.$$

Let $I$ be a nontrivial initial interval. Let us show that $I = \downarrow\mathbb{Z}(x)^\beta$ for some $x \in I$ and $\beta < \alpha$. Let $\beta < \alpha$ be the least such that

$$x <_\beta y \text{ for some } x \in I \text{ and } y \notin I. \tag{$*$}$$

Then, search for $x_0, y_0 \in \mathbb{Z}^\alpha$ so that $(*)$ holds and $|(x_0(\beta), y_0(\beta))_\mathbb{Z}| = n \in \mathbb{N}$ is $\omega$-least (in this case $n = 0$). We claim that $I = \downarrow\mathbb{Z}(x_0)^\beta$.

Let $z \in I$. If $z \leq x_0$, then $z \in L(\preceq x_0) \subseteq \downarrow\mathbb{Z}(x_0)^\beta$. Assume $x_0 < z < y_0$. It follows that $x_0(\beta) \leq_\mathbb{Z} z(\beta) \leq_\mathbb{Z} y_0(\beta)$. The case $z(\beta) = y_0(\beta)$ contradicts the minimality of $\beta$. Then $x_0(\beta) = z(\beta)$ and hence $z \in \mathbb{Z}(x_0)^\beta$. For the converse, if $z \leq x_0$, then $z \in I$. So let $z \in \mathbb{Z}(x_0)^\beta$ with $x_0 < z$. By definition, $x \prec_\gamma z$ for some $\gamma < \beta$. Then, by the minimality of $\beta$, $z \in I$. □

## 5.3 Hausdorff's rank

We first review the notion of Hausdorff's rank in the framework of reverse mathematics. Then we provide a proof of Theorem 5.1.3 by following the hint in [Clo89, Theorem 16].

**Definition 5.3.1** ($\mathsf{ATR}_0$)**.** Let $L$ be a linear order and $\alpha$ be a well-order. By arithmetical transfinite recursion on $\beta \leq \alpha$, we define a binary relation $\sim_\beta$ on $L$ by letting $x \sim_\beta y$ if and only if $x = y$ or for some $\gamma < \beta$ there exists a finite set $F \subseteq L$ such that for every $z \in [x, y]_L$ there is $w \in F$ with $z \sim_\gamma w$.

In the second case, we say that the interval $[x, y]_L$ is *finite modulo* $\gamma$ and define the *cardinality of* $[x, y]_L$ *modulo* $\gamma$ as the least cardinality of a witness $F$.

**Lemma 5.3.2** ($\mathsf{ATR}_0$)**.** *Let $L$ be a linear order and $\alpha$ be a well-order. Then for all $\beta \leq \alpha$ the following hold:*

- *$x \sim_\beta y$ imply $y \sim_\beta x$;*
- *if $x \sim_\beta y$ and $x <_L z <_L y$, then $x \sim_\beta z$ and $z \sim_\beta y$;*
- *the relation $\sim_\beta$ is an equivalence relation;*
- *every equivalence class of $\sim_\beta$ is an interval of $L$.*

*Proof.* Straightforward. □

**Definition 5.3.3** ($\mathsf{ATR}_0$)**.** Let $L$ be a linear order and $\alpha$ a well-order. We say that $L$ has *Hausdorff rank at most* $\alpha$ and write $\mathrm{rk}_{\mathrm{H}}(L) \leq \alpha$ if $x \sim_\alpha y$ for all $x, y \in L$.

*Proof of Theorem 5.1.3.* Let $L$ be a scattered linear order. By Theorem 5.1.2, let $\alpha$ be a well-order such that $\mathrm{rk}_{\mathrm{H}}(L) \leq \alpha$. By arithmetical transfinite recursion we define for all $\beta \leq \alpha$ a function $f_\beta \colon L \to \mathbb{Z}^\beta = \mathbb{Z}(0)^\beta$ such that:

$(*)$ $x <_L y$ implies $f_\beta(x) < f_\beta(y)$ for all $x, y \in L$ with $x \sim_\beta y$.

For $\beta = \alpha$, we obtain an embedding of $L$ into $\mathbb{Z}^\alpha$ as desired.

Let $\beta \leq \alpha$ and suppose we have defined $f_\gamma$ for all $\gamma < \beta$. Let $x \in L$ be given and $x_\beta \in L$ be $\omega$-least such that $x \sim_\beta x_\beta$. If $x = x_\beta$, let $f_\beta(x) = 0$. Otherwise, let $\gamma < \beta$ be least such that the interval $[x, x_\beta]_L$ is finite modulo $\gamma$ and let $n >_{\mathbb{Z}} 0$ be its cardinality modulo $\gamma$. Now let

$$f_\beta(x) = \begin{cases} \langle(\gamma, n)\rangle^\frown f_\gamma(x) & \text{if } x_\beta <_L x; \\ \langle(\gamma, -n\rangle^\frown f_\gamma(x) & \text{otherwise.} \end{cases}$$

It remains to prove that every $f_\beta$ is a function from $L$ to $\mathbb{Z}^\beta$ and satisfies $(*)$. We prove this by arithmetical transfinite induction on $\beta \leq \alpha$. Suppose the properties hold for all $\gamma < \beta$. It immediately follows that $f_\beta$ is a function from $L$ to $\mathbb{Z}^\beta$.

Now, let $x \sim_\beta y$ with $x <_L y$. By minimality, $z = x_\beta = y_\beta$. There are several cases to consider, but the only interesting ones are $x <_L y <_L z$ and $z <_L x <_L y$. We consider the case $x <_L y <_L z$ and leave the others to the reader.

Notice that if $[x, z]_L$ is finite modulo $\gamma$ via $F$, then so is $[y, z]_L$.

Let,

$$f_\beta(x) = \langle(\gamma_0, -n)\rangle^\frown f_{\gamma_0}(x) \text{ and } f_\beta(y) = \langle(\gamma_1, -m)\rangle^\frown f_{\gamma_1}(y),$$

where $\gamma_0, \gamma_1 < \beta$ and $n, m >_\mathbb{Z} 0$ are defined as above.

By the minimality of $\gamma_1$, $\gamma_1 \leq \gamma_0$. If $\gamma_1 < \gamma_0$, then by definition $f_\beta(x) < f_\beta(y)$. Suppose $\gamma = \gamma_0 = \gamma_1$. Therefore, by minimality again, $m \leq_\mathbb{Z} n$.

If $m <_\mathbb{Z} n$, then $f_\beta(x) < f_\beta(y)$ by definition. Then suppose $m = n$. We claim that $x \sim_\gamma y$ so that the induction hypothesis applies yielding $f_\gamma(x) < f_\gamma(y)$ and thus

$$f_\beta(x) = \langle(\gamma, -n)\rangle^\frown f_\gamma(x) < \langle(\gamma, -n)\rangle^\frown f_\gamma(y) = f_\beta(y).$$

Let $F \subseteq L$ be of cardinality $n$ such that $[x, z]_L$ is finite modulo $\gamma$ via $F$. As noted before, $F$ witnesses that $[y, z]_L$ is finite modulo $\gamma$ too. Since $n = m$, for all $w \in F$ there exists $u \in [y, z]_L$ such that $u \sim_\gamma w$. Now take $w \in F$ such that $x \sim_\gamma w$ and $u \in [y, z]_L$ such that $u \sim_\gamma w$. Since $\sim_\gamma$ is an equivalence relation, we have $x \sim_\gamma u$. The claim now follows from the fact that the equivalence classes are intervals. □

As a consequence of Lemma 5.2.8, we obtain the following:

**Corollary 5.3.4.** *Over* $\mathsf{ACA}_0$*, the following are equivalent:*

(1) $\mathsf{ATR}_0$*;*

(2) *every scattered linear order embeds into* $\mathbb{Z}^\alpha$ *for some well-order* $\alpha$*.*

*Proof.* (1) $\Rightarrow$ (2) is Theorem 5.1.3. (2) $\Rightarrow$ (1) follows from Theorem 4.3.3 and Lemma 5.2.8 since $\mathsf{WKL}_0$ (and hence $\mathsf{ACA}_0$) shows that if a partial order has countably many initial intervals then any other partial order embeddable into it has countable many initial intervals (see Theorem 4.4.4). □

## 5.4 A theorem in $\mathsf{ATR}_0$

In this section, we prove Theorem 5.1.4. The proof comes down to show that every embedding of $\omega^\alpha$ into $\mathbb{Z}^\alpha$ is cofinal.

**Lemma 5.4.1** ($\mathsf{ACA}_0$). *If $\alpha$ is a well-order, then every embedding $f\colon \omega^\alpha \to \mathbb{Z}^\alpha$ is cofinal.*

*Proof.* Let $f$ be an embedding of $\omega^\alpha$ into $\mathbb{Z}^\alpha$. By arithmetical transfinite induction on $\beta < \alpha$, we prove that for all $\beta \leq \alpha$, $\delta < \omega^\alpha$ and $x \in \mathbb{Z}^\alpha$, if $f$ maps $[\delta, \delta + \omega^\beta)$ into $\mathbb{Z}(x)^\beta$, then the restriction of $f$ to $[\delta, \delta + \omega^\alpha)$ is cofinal on $\mathbb{Z}(x)^\beta$.

Let $\beta < \alpha$ and assume the property holds for all $\gamma < \beta$. Suppose now that $f$ maps $[\delta, \delta+\omega^\beta)$ noncofinally into $\mathbb{Z}(x)^\beta$. Then there exist $y, z \in \mathbb{Z}(x)^\beta$ such that $y \leq f(\lambda) \leq z$ for all $\lambda \in [\delta, \delta+\omega^\beta)$. Let $\gamma < \beta$ be so that $y <_\gamma z$. It follows that, for all $\lambda \in [\delta, \delta+\omega^\beta)$,

i) $y(\gamma') = f(\lambda)(\gamma') = z(\gamma')$ for all $\gamma < \gamma' < \alpha$ and

ii) $y(\gamma) \leq_{\mathbb{Z}} f(\lambda)(\gamma) \leq_{\mathbb{Z}} z(\gamma)$.

By (ii), there must be $\lambda \in [\delta, \delta + \omega^\beta)$ such that $n = f(\lambda)(\gamma)$ is $\mathbb{Z}$-greatest. Let $y = f(\lambda)$. By (i) and the maximality of $n$, $f$ maps $[\lambda, \lambda + \omega^\beta)$ into $\mathbb{Z}(y)^\gamma$. Therefore, $f$ maps $[\lambda, \lambda + \omega^\gamma)$ noncofinally into $\mathbb{Z}(y)^\gamma$, contrary to the induction hypothesis.

For $\beta = \alpha$, $\delta = 0$ and $x = 0$ we obtain the conclusion. □

**Theorem 5.4.2** ($\mathsf{ATR}_0$). *If $L$ is a scattered linear order, then there exists a well-order which does not embed into $L$.*

*Proof.* Let $L$ be a scattered linear order. By Theorem 5.1.3, there is a well-order $\alpha$ such that $L$ embeds into $\mathbb{Z}^\alpha$. By Lemma 5.4.1, $\omega^\alpha + 1$ is not embeddable into $\mathbb{Z}^\alpha$, and thus does not embed into $L$. On the other hand, by Theorem 5.2.2, $\omega^\alpha + 1$ is a well-order. This completes the proof. □

## 5.5 Hausdorff's classification theorem for scattered linear orders

We aim to study the following classification theorem by Hausdorff.

**Theorem 5.5.1** ([Hau08]). *The class of countable scattered linear orders is the least class which contains the empty set, singletons and is closed under lexicographic sums along $\mathbb{Z}$.*

This theorem does not translate directly in second-order arithmetic. Our formalization is quite standard (see for instance the coding of Borel sets in [Sim09]): we use well-founded trees to code the construction of a given scattered linear order.

**Definition 5.5.2** ($\mathsf{RCA}_0$)**.** A *code* (*for a countable scattered linear order*) is a well-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$.

If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree, let $E(T)$ be the set of end-nodes of $T$. Notice that $E(T)$ exists in $\mathsf{ACA}_0$. Nonetheless, in $\mathsf{RCA}_0$ we can write $\sigma \in E(T)$ as a shorthand for $\sigma \in T \wedge (\forall n)\sigma^\frown\langle n\rangle \notin T$.

**Definition 5.5.3** ($\mathsf{ATR}_0$)**.** Let $T$ be a code. For $\sigma \in T$, let $L_\sigma$ be the restriction of $\mathbb{Z}$ to $\{n \in \mathbb{N} \colon \sigma^\frown\langle n\rangle \in T\}$. By arithmetical recursion on $\sigma \in T$, we define a linear order $L^\sigma$ by:

$$L^\sigma = \begin{cases} \{0\} & \text{if } \sigma \in E(T); \\ \sum_{n \in L_\sigma} L^{\sigma^\frown\langle n\rangle} & \text{otherwise.} \end{cases}$$

Finally, we set $L(T) = L^{\langle\rangle}$.

We then formalize Hausdorff's theorem as follows.

**Theorem 5.5.4.** *Let $L$ be linear order. Then $L$ is scattered if and only if there exists a code $T$ such that $L$ is isomorphic to $L(T)$.*

We will show that Theorem 5.5.4 is provable in $\mathsf{ATR}_0$. Notice that we cannot reverse this theorem to $\mathsf{ATR}_0$, because the statement does not make sense in a weaker system. However, given a code $T$, it is possible to define, this time in $\mathsf{ACA}_0$, another linear order which, provably in $\mathsf{ATR}_0$, is isomorphic to $L(T)$. With this definition in hand, we state Hausdorff's theorem in $\mathsf{ACA}_0$ and show that it is equivalent to $\mathsf{ATR}_0$ over $\mathsf{ACA}_0$.

### 5.5.1 Hausdorff's theorem in $\mathsf{ACA}_0$

**Definition 5.5.5** ($\mathsf{ACA}_0$)**.** For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, let $\sigma \leq \tau$ if and only if $\sigma \supseteq \tau$ or $n <_{\mathbb{Z}} m$, where $n, m$ are unique such that $\eta^\frown\langle n\rangle \sqsubseteq \sigma$ and $\eta^\frown\langle m\rangle \sqsubseteq \tau$.

If $T$ is a code, let $E(T)$ be linearly ordered by $\leq$.

**Lemma 5.5.6** ($\mathsf{ATR}_0$)**.** *If $T$ is a code, then $L(T) \cong E(T)$.*

*Proof.* For all $\sigma \in T$, let $T_\sigma = \{\tau \colon \sigma^\frown\tau \in T\}$. By arithmetical recursion on $\sigma \in T$, we define an isomorphism $f_\sigma \colon L_\sigma \to E(T_\sigma)$. The case $\sigma \in E(T)$ is immediate to handle, since $L_\sigma = \{0\}$ and $E(T) = \{\langle\rangle\}$. For $\sigma \in T \setminus E(T)$, it is enough to notice that $E(T_\sigma) \cong \sum_{\{n \in \mathbb{Z} \colon \sigma^\frown\langle n\rangle \in T\}} E(T_{\sigma^\frown\langle n\rangle})$ and the isomorphism is arithmetically definable (uniformly in $\sigma$). $\square$

We then consider the following formulation of Hausdorff's theorem.

**Theorem 5.5.7.** *Let $L$ be linear order. Then $L$ is scattered if and only $L$ is isomorphic to $E(T)$ for some code $T$.*

We will show that the left-to-right direction of Theorem 5.5.7 reverses to $\mathsf{ATR}_0$ over $\mathsf{ACA}_0$, while the right-to-left direction is already provable in $\mathsf{ACA}_0$ (Theorem 5.5.9).

### 5.5.2 Proofs in $\mathsf{ACA}_0$ and equivalence with $\mathsf{ATR}_0$

**Lemma 5.5.8** ($\mathsf{ACA}_0$)**.** *If $T$ is a code, then $E(T)$ is scattered.*

*Proof.* By way of contradiction, suppose $T$ is a code and $E(T)$ is not scattered. Let $f\colon \mathbb{Q} \to E(T)$ be an embedding and for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let $E_\sigma = \{\tau \in E(T) \colon \sigma \sqsubseteq \tau\}$. By $\mathbf{\Pi}^0_1$ transfinite induction on $\sigma \in T$ (see Lemma 5.2.6), we prove

$$(\forall \sigma \in T)(\forall i, j \in \mathbb{Q})(i \neq j \implies f(i) \notin E_\sigma \vee f(j) \notin E_\sigma).$$

Hence, for $\sigma = \langle\rangle$, we have a contradiction. If $\sigma \in E(T)$, then $E_\sigma = \{\sigma\}$, and so the property is obviously satisfied. Now, let $\sigma \in T \setminus E(T)$. Notice that $E_\sigma = \bigcup_{n\in\mathbb{Z}} E_{\sigma^\frown\langle n\rangle}$. Let $i, j \in \mathbb{Q}$ so that $i <_{\mathbb{Q}} j$ and suppose $f(i), f(j) \in E_\sigma$. Since $f$ is order preserving, $f(i) < f(j)$ and so $f(i) \in E_{\sigma^\frown\langle n\rangle}$ and $f(j) \in E_{\sigma^\frown\langle m\rangle}$ for some $n \leq_{\mathbb{Z}} m$ with $\sigma^\frown\langle n\rangle, \sigma^\frown\langle m\rangle \in T$. Since any finite set is scattered, there exists $i_0, j_0 \in \mathbb{Q}$ and $k \in \mathbb{Z}$ with $\sigma^\frown\langle k\rangle \in T$ such that $i \leq_{\mathbb{Q}} i_0 <_{\mathbb{Q}} j_0 \leq_{\mathbb{Q}} j$, $n \leq_{\mathbb{Z}} k \leq_{\mathbb{Z}} m$ and $f(i_0), f(j_0) \in E_{\sigma^\frown\langle k\rangle}$, contrary to the induction hypothesis. $\square$

As a corollary, we obtain the following:

**Theorem 5.5.9** ($\mathsf{ACA}_0$)**.** *Let $L$ be a linear order. If there exists a code $T$ such that $L \cong E(T)$, then $L$ is scattered.*

*Proof.* Immediate from the above lemma since isomorphisms preserve (provably in $\mathsf{RCA}_0$) scattered linear orders. $\square$

We now show the hard direction of Theorem 5.5.7.

**Theorem 5.5.10** ($\mathsf{ATR}_0$)**.** *Every scattered linear order is isomorphic to $E(T)$ for some code $T$.*

*Proof.* Let $L$ be a countable scattered linear order. It clearly suffices to show that there exists a well-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $L$ is embeddable in $E(T)$.

By Theorem 5.1.3, $L$ embeds into $\mathbb{Z}^\alpha$ for some well-order $\alpha$. We may therefore assume $L = \mathbb{Z}^\alpha$. If $x \in \mathbb{Z}^\alpha$, let $\deg(x)$ be the largest $\beta < \alpha$ such that $x(\beta) \neq 0$. For $\beta \leq \alpha$, we also define $\mathbb{Z}^\beta = \{x \in \mathbb{Z}^\alpha \colon \deg(x) < \beta\}$.

By arithmetical transfinite recursion on $\beta \leq \alpha$, we define for all $\beta \leq \alpha$

- a tree $T_\beta \subseteq \mathbb{N}^{<\mathbb{N}}$, a function $r_\beta \colon T_\beta \to \beta + 1$ and
- a function $f_\beta \colon \mathbb{Z}^\beta \to E(T_\beta)$.

Case $\beta = 0$. Let $T_0 = \{\langle\rangle\}$, $r_0(\langle\rangle) = 0$ and $f_0(0) = \langle\rangle$.

Case $\beta+1$. Define $T_{\beta+1} = \{\langle\rangle\} \cup \{\langle n\rangle^\frown\sigma \colon n \in \mathbb{N} \wedge \sigma \in T_\beta\}$. We define $r_{\beta+1}$ by letting $r_{\beta+1}(\langle\rangle) = \beta + 1$ and $r_{\beta+1}(\langle n\rangle^\frown\sigma) = r_\beta(\sigma)$. We then define $f_{\beta+1} \colon \mathbb{Z}^{\beta+1} \to E(T_{\beta+1})$ by letting

$$f_{\beta+1}(x) = \begin{cases} \langle 0\rangle^\frown f_\beta(x) & \text{if } x \in \mathbb{Z}^\beta; \\ \langle x(\beta)\rangle^\frown f_\beta(x \restriction \beta) & \text{if } \deg(x) = \beta. \end{cases}$$

Case $\lambda$ limit. Let $\gamma_0 < \gamma_1 < \gamma_2 < \dots$ be cofinal in $\lambda$ and $(n_k), (m_k)$ be two sequences in $\mathbb{Z}$ such that $\dots <_\mathbb{Z} n_1 <_\mathbb{Z} n_0 = m_0 <_\mathbb{Z} m_1 <_\mathbb{Z} \dots$ Define

$$T_\lambda = \{\langle\rangle\} \cup \{\langle n\rangle^\frown\sigma \colon (\exists k)(n \in \{n_k, m_k\} \wedge \sigma \in T_{\gamma_k})\}.$$

We define $r_\lambda$ by letting $r_\lambda(\langle\rangle) = \lambda$ and $r_\lambda(\langle n\rangle^\frown\sigma) = r_{\gamma_k}(\sigma)$, where $n \in \{n_k, m_k\}$. We finally define $f_\lambda \colon \mathbb{Z}^\lambda \to E(T_\lambda)$ by letting for $x \in \mathbb{Z}^\gamma$ and $\beta = \deg(x)$:

$$f_\lambda(x) = \begin{cases} \langle n_0\rangle^\frown f_{\gamma_0}(x) & \text{if } \beta < \gamma_0; \\ \langle n_{k+1}\rangle^\frown f_{\gamma_{k+1}}(x) & \text{if } \gamma_k \leq \beta < \gamma_{k+1} \text{ and } x(\beta) <_\mathbb{Z} 0; \\ \langle m_{k+1}\rangle^\frown f_{\gamma_{k+1}}(x) & \text{if } \gamma_k \leq \beta < \gamma_{k+1} \text{ and } x(\beta) >_\mathbb{Z} 0; \end{cases}$$

It is not difficult to verify by arithmetical transfinite induction on $\beta \leq \alpha$ that for all $\beta \leq \alpha$,

- if $\sigma, \tau \in T_\beta$ and $\sigma \supset \tau$ then $r_\beta(\sigma) > r_\beta(\tau)$ and hence $T_\beta$ is well-founded;
- $f_\beta$ is an embedding of $\mathbb{Z}^\beta$ into $E(T_\beta)$.

For $\beta = \alpha$, we have the desired conclusion. $\square$

**Corollary 5.5.11** ($\mathsf{ATR}_0$)**.** *Every scattered linear order is isomorphic to $L(T)$ for some code $T$.*

*Proof.* Immediate from Lemma 5.5.6 and Theorem 5.5.10. □

**Theorem 5.5.12.** *Over $\mathsf{ACA}_0$, the following are equivalent:*

(1) $\mathsf{ATR}_0$;

(2) *every scattered linear order is isomorphic to $E(T)$ for some code $T$.*

*Proof.* (1) $\Rightarrow$ (2) is Theorem 5.5.10. For (2) $\Rightarrow$ (1), it is enough to prove by Theorem 4.3.3 and Corollary 4.4.4 that $E(T)$ has countably many initial intervals for any code $T$.

Let $T$ be a code and $L = E(T)$. For $\sigma \in T$, let $I_\sigma = \{\tau \in L \colon \tau \leq \sigma\}$. We claim that

$$\mathcal{I}(L) = \{\emptyset\} \cup \{I_\sigma \colon \sigma \in T\} \cup \{L\}.$$

Clearly, every $I_\sigma$ is an initial interval. Let $I \subseteq L$ be a nontrivial initial interval. By $\mathbf{\Pi}^0_1$ transfinite induction, let $\sigma \in T$ be minimal such that

$$(\exists \tau_0, \tau_1 \in L)(\tau_0 \in I \wedge \tau_1 \notin I \wedge \sigma = \tau_0 \cap \tau_1),$$

where $\tau_0 \cap \tau_1$ is the longest common initial segment of $\tau_0$ and $\tau_1$. By $\mathbf{\Pi}^0_1$ induction, there exists $n <_{\mathbb{Z}} m$ such that $(n, m)_{\mathbb{Z}} = \emptyset$ and

$$(\exists \tau_0, \tau_1 \in L)(\tau_0 \in I \wedge \tau_0 \supseteq \sigma^\frown\langle n\rangle \wedge \tau_1 \notin I \wedge \tau_1 \supseteq \sigma^\frown\langle m\rangle).$$

Let us prove $I = I_{\sigma^\frown\langle n\rangle}$. Fix $\tau_0$ and $\tau_1$ as above.

Let $\tau \in I$. We may assume $\tau_0 < \tau < \tau_1$. It follows the definition that either $\tau \supseteq \sigma^\frown\langle n\rangle$ or $\tau \supseteq \sigma^\frown\langle m\rangle$. The minimality of $\sigma$ leaves out the second case. Then $\tau \leq \sigma^\frown\langle n\rangle$.

Let $\tau \in I_{\sigma^\frown\langle n\rangle}$, that is $\tau \in L$ and $\tau \leq \sigma^\frown\langle n\rangle$. If $\tau \leq \tau_0$, we are done. Hence, assume $\tau_0 < \tau$. By the definition of $\leq$, it follows that $\tau_0 \cap \tau \supseteq \sigma^\frown\langle n\rangle$. By the minimality of $\sigma$ again, $\tau \in I$. □

# 6

# Hausdorff-like theorems

## 6.1 Introduction

Let us recall Hausdorff's theorem for scattered linear orders.

**Theorem 6.1.1** ([Hau08]). *The class of countable scattered linear orders is the least class which contains the empty set, singletons and is closed under lexicographic sums along $\mathbb{Z}$.*

In this chapter, we study the reverse mathematics of two classification theorems, which are the analogue of Hausdorff's theorem for the class of scattered FAC partial orders (Theorem 6.1.3) and the class of countable FAC partial orders (Theorem 6.1.4) respectively.

Recall the following definitions from subsection 1.5.1.

**Definition 6.1.2** ($\mathsf{RCA}_0$). Let $(P, \preceq)$ be a partial order.

- The *inverse* (or *reverse*) of $P$ is $P^* = (P, \succeq)$;
- A *restriction* of $P$ is $S \subseteq P$ equipped with the ordering induced by $P$, namely $x \leq_S y$ if and only if $x \preceq y$ for all $x, y \in S$;
- An *extension* of $P$ is a partial order $P' = (P, \preceq')$ such that $x \preceq y$ implies $x \preceq' y$ for all $x, y \in P$.

**Theorem 6.1.3** ([AB99]). *Let $\mathcal{B}$ be the class of wpo's and reverse wpo's. The class of* scattered FAC *partial orders is the least class which contains $\mathcal{B}$ and is closed under extensions and sums with index set in $\mathcal{B}$.*

**Theorem 6.1.4** (§7, [ABC$^+$12]). *Let $\mathcal{B}$ be the class of countable partial orders which are either wpo's, reverse wpo's or linear orders. The class of* countable FAC *partial orders is the least class which contains $\mathcal{B}$ and is closed under extensions and sums with index set in $\mathcal{B}$.*

As for Hausdorff's theorem for scattered linear orders we need to formalize each of the above theorems in second-order arithmetic. In other words, given $\mathcal{B}$, we have to define the least class $\mathcal{C}(\mathcal{B})$ which contains $\mathcal{B}$ and is closed under extensions and sums over $\mathcal{B}$. We do this by means of well-founded trees labeled with partial orders in the class $\mathcal{B}$ (see section 6.2).

It turns out that the easy direction of each theorem is provable in $\mathsf{ACA}_0$. By the easy direction we mean the statement "if $\mathcal{B}$ is the class of wpo's and reverse wpo's (wpo's, reverse wpo's and linear orders) and $P \in \mathcal{C}(\mathcal{B})$, then $P$ is scattered FAC (FAC)". For the hard direction, we provide a proof in $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$. This upper bound is not the best possible because each statement is $\mathbf{\Pi}^1_3$ and by standard arguments a $\mathbf{\Pi}^1_3$ statement cannot imply $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$. Therefore we do not completely succeed in answering the reverse mathematics question about the strength of these theorems and further investigation needs to be done.

## 6.2 Codes

In general, it is not difficult to show that a partial order obtained by iterating sums and extensions can be equivalently obtained by iterating only sums and then applying exactly one extension at the end. We code iterated sums as follows:

**Definition 6.2.1** ($\mathsf{ATR}_0$)**.** A *code* is a well-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ labeled with partial orders $\{P_\sigma \colon \sigma \in T\}$ such that $P_\sigma$ is a partial order on $\{x \colon \sigma^\frown\langle x\rangle \in T\}$ for every interior-node $\sigma \in T$.

Given a code $\{P_\sigma \colon \sigma \in T\}$, by transfinite recursion on $\sigma \in T$ we define the partial order $P^\sigma$ by letting:

$$P^\sigma = \begin{cases} P_\sigma & \text{if } \sigma \text{ is an end-node;} \\ \sum_{x \in P_\sigma} P^{\sigma^\frown\langle x\rangle} & \text{otherwise.} \end{cases}$$

Finally, we set $P(T) = P^{\langle\rangle}$.

The above inductive definition can be "spelled out" as follows.

**Definition 6.2.2** ($\mathsf{ACA}_0$)**.** Let $\{P_\sigma \colon \sigma \in T\}$ be a code. We define $\sum_{\sigma \in T} P_\sigma$ to be the partial order $\sum_{\sigma \in E(T)} P_\sigma$, where $E(T)$ (the set of end-nodes of $T$) is ordered by letting $\sigma \leq \tau$ if and only if $\sigma = \tau$ or $x <_{P_\mu} y$, where $\mu^\frown\langle x\rangle \subseteq \sigma$ and $\mu^\frown\langle y\rangle \subseteq \tau$.

This definition in $\mathsf{ACA}_0$ can be thought of as an order-theoretic operation that given a code $\{P_\sigma : \sigma \in T\}$ outputs a new partial order $\sum_{\sigma\in T} P_\sigma$. We now show that the two definitions are equivalent:

**Lemma 6.2.3** ($\mathsf{ATR}_0$)**.** *Let $\{P_\sigma : \sigma \in T\}$ be a code. Then $P(T) \cong \sum_{\sigma\in T} P_\sigma$.*

*Proof.* For all $\sigma \in T$, let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $P_{\sigma,\tau} = P_{\sigma^\frown\tau}$ for all $\tau \in T_\sigma$. By arithmetical transfinite recursion on $\sigma \in T$, it is not difficult to define an isomorphism $f_\sigma : P^\sigma \to \sum_{\tau\in T_\sigma} P_{\sigma,\tau}$. For $\sigma = \langle\rangle$ we have the desired isomorphism. $\square$

As a consequence, we can set aside the inductive definition and use the more convenient, although less intuitive, definition in $\mathsf{ACA}_0$.

**Definition 6.2.4** ($\mathsf{ACA}_0$)**.** Let $\mathcal{B}$ be a class of partial orders and $P$ be a partial order.

- We say that $\{P_\sigma : \sigma \in T\}$ is a $\mathcal{B}$-code if $P_\sigma \in \mathcal{B}$ for every $\sigma \in T$.
- We write $P \in \mathcal{C}(\mathcal{B})$ if there exists a $\mathcal{B}$-code $\{P_\sigma : \sigma \in T\}$ such that $P$ is isomorphic to an extension of $\sum_{\sigma\in T} P_\sigma$. In other words, there exists an order-preserving map from $\sum_{\sigma\in T} P_\sigma$ onto $P$.

**Observation 6.2.5.** If $\mathcal{B}$ is $\mathbf{\Pi}^1_1$, then $\mathcal{C}(\mathcal{B})$ is $\mathbf{\Sigma}^1_2$. Besides, the statement "every scattered FAC (FAC) partial order is in $\mathcal{C}(\mathcal{B})$" is $\mathbf{\Pi}^1_3$.

## 6.3 Closure properties

Let $\mathcal{B}$ be a class of partial orders. We show that codes "are closed" under inverses, restrictions and sums over $\mathcal{B}$.

**Remark 6.3.1** ($\mathsf{RCA}_0$)**.** Let $\mathcal{B}$ be the class of wpo's and reverse wpo's (wpo's, reverse wpo's and linear orders). Then $\mathcal{B}$ is closed under inverses and restrictions.

**Lemma 6.3.2** ($\mathsf{ACA}_0$)**.** *Suppose $\mathcal{B}$ is closed under inverses and let $\langle P_\sigma : \sigma \in T\rangle$ be a $\mathcal{B}$-code. Then $\langle P^*_\sigma : \sigma \in T\rangle$ is a $\mathcal{B}$-code and $(\sum_{\sigma\in T} P_\sigma)^* = \sum_{\sigma\in T} P^*_\sigma$.*

*Proof.* Straightforward. Notice that, since $\mathcal{B}$ is closed under inverses, $\langle P^*_\sigma : \sigma \in T\rangle$ is a $\mathcal{B}$-code. $\square$

**Lemma 6.3.3** ($\mathsf{ACA}_0$)**.** *Suppose $\mathcal{B}$ is closed under restrictions and let $\{P_\sigma : \sigma \in T\}$ be a $\mathcal{B}$-code. If $S$ is a restriction of $\sum_{\sigma\in T} P_\sigma$, then $S = \sum_{\sigma\in T} S_\sigma$ for some $\mathcal{B}$-code $\{S_\sigma : \sigma \in T\}$.*

*Proof.* Let $\langle P_\sigma : \sigma \in T\rangle$ and $S$ be as above. Define for $\sigma \in T$:

$$S_\sigma = \begin{cases} \{x \in P_\sigma : (\sigma, x) \in S\} & \text{if } \sigma \text{ is an end-node,} \\ P_\sigma & \text{otherwise.} \end{cases}$$

As $\mathcal{B}$ is closed under restrictions, $\langle S_\sigma : \sigma \in T\rangle$ is a $\mathcal{B}$-code. It is immediate to verify that $S = \sum_{\sigma \in T} S_\sigma$. □

**Lemma 6.3.4** ($\mathsf{ACA}_0$)**.** *Let $B \in \mathcal{B}$ and $\{P_{x,\sigma} : \sigma \in T_x\}$ be a $\mathcal{B}$-code for each $x \in B$. Then $P = \sum_{x \in B} P_x \cong \sum_{\sigma \in T} P_\sigma$ for some $\mathcal{B}$-code $\{P_\sigma : \sigma \in T\}$, where $P_x = \sum_{\sigma \in T_x} P_{x,\sigma}$ for all $x \in B$.*

*Proof.* Define a $\mathcal{B}$-code $\{P_\sigma : \sigma \in T\}$ by letting:

- $T = \{\langle\rangle\} \cup \{\langle x\rangle^\frown\sigma : x \in B \wedge \sigma \in T_x\}$;
- $P_{\langle\rangle} = B$ and $P_{\langle x\rangle^\frown\sigma} = P_{x,\sigma}$.

Clearly we have defined a $\mathcal{B}$-code. We next define an isomorphism $f$ between $P = \sum_{x \in B} P_x$ and $\sum_{\sigma \in T} P_\sigma$ by letting $f(x, (\sigma, y)) = (\langle x\rangle^\frown\sigma, y)$ for all $x \in B$, $\sigma \in E(T_x)$ and $y \in P_{x,\sigma}$. It is not difficult to check that $f$ is an isomorphism. □

**Lemma 6.3.5** ($\mathsf{ACA}_0$)**.** *Let $\mathcal{B}$ be a class of partial orders. Then $\mathcal{C}(\mathcal{B})$ is closed under extensions. Moreover, if $\mathcal{B}$ is closed under under inverses and restrictions, then $\mathcal{C}(\mathcal{B})$ is closed under inverses and restrictions as well.*

*Proof.* Closure under extensions is trivial because if $f : P \to P'$ is order-preserving and $P''$ extends $P'$ then $f$ is still order-preserving with respect to $P$ and $P''$.

Suppose $\mathcal{B}$ is closed under under inverses and restrictions. Suppose $P \in \mathcal{C}(\mathcal{B})$ and $S \subseteq P$. To show that $P^* \in \mathcal{C}(\mathcal{B})$, notice that an order-preserving map from $\sum_{\sigma \in T} P_\sigma$ onto $P$ is order-preserving with respect to the inverse of $\sum_{\sigma \in T} P_\sigma$ and $P^*$. By Lemma 6.3.2 $(\sum_{\sigma \in T} P_\sigma)^*$ has a $\mathcal{B}$-code. Similarly, by using Lemma 6.3.3, one shows that $S \in \mathcal{C}(\mathcal{B})$. □

**Lemma 6.3.6** ($\mathbf{\Sigma}^1_2$-$\mathsf{AC}_0$)**.** *Let $\mathcal{B}$ be a $\mathbf{\Pi}^1_1$ class of partial orders. Then $\mathcal{C}(\mathcal{B})$ is closed under sums over $\mathcal{B}$.*

*Proof.* Let $P = \sum_{x \in B} P_x$ be a sum such that $B \in \mathcal{B}$ and $P_x \in \mathcal{C}(\mathcal{B})$ for all $x \in B$. As noted before, $\mathcal{C}(\mathcal{B})$ is $\mathbf{\Sigma}^1_2$ and so we can apply $\mathbf{\Sigma}^1_2$ choice to fix a $\mathcal{B}$-code for every $P_x$. Finally, use Lemma 6.3.4 to obtain the conclusion. □

Recall that $\mathbf{\Sigma}^1_2$-$\mathsf{AC}_0$ is equivalent to $\mathbf{\Delta}^1_2$-$\mathsf{CA}_0$ and to $\mathbf{\Pi}^1_2$ separation. In particular, it is available in $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$.

**Corollary 6.3.7** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *If $\mathcal{B}$ is $\mathbf{\Pi}^1_1$ class of partial order closed under inverses and restrictions (such as the class of scattered FAC and the class of FAC partial orders), then $\mathcal{C}(\mathcal{B})$ is closed under extensions, inverses, restrictions and sums over $\mathcal{B}$.*

## 6.4 Proofs in $\mathsf{ACA}_0$

**Theorem 6.4.1** ($\mathsf{ACA}_0$)**.** *If $\mathcal{B}$ consists of FAC partial orders and $P \in \mathcal{C}(\mathcal{B})$, then $P$ is FAC.*

*Proof.* Since isomorphisms preserve FAC partial orders and every extension of a FAC partial order is FAC, it suffices to prove that for every $\mathcal{B}$-code $\langle P_\sigma \colon \sigma \in T\rangle$ the partial order $\sum_{\sigma\in T} P_\sigma$ is FAC. Provably in $\mathsf{ACA}_0$, every sum of FAC partial orders along a FAC partial order is FAC (see Lemma 1.5.1). It thus suffices to show that $E(T)$ is FAC. Suppose not and let $A \subseteq E(T)$ be an infinite antichain.

Form the tree $S = \{\tau \in T \colon (\exists \sigma \in A)\tau \subseteq \sigma\}$. It is easy to see that $S$ is finitely branching, otherwise there would be $\tau \in S$ such that $P_\tau$ is not FAC. Then $S$ is well-founded (being a subtree of $T$) and finitely branching. By König's lemma, $S$ is finite, contradicting $A \subseteq S$. □

**Theorem 6.4.2** ($\mathsf{ACA}_0$)**.** *If $\mathcal{B}$ consists of scattered FAC partial orders and $P \in \mathcal{C}(\mathcal{B})$, then $P$ is scattered FAC.*

*Proof.* Let $\langle P_\sigma \colon \sigma \in T\rangle$ be a $\mathcal{B}$-code such that $P$ is isomorphic to an extension of $\sum_{\sigma\in T} P_\sigma$. By the proof of the previous theorem, $\sum_{\sigma\in T} P_\sigma$ is FAC and so $P$ is FAC. Since provably in $\mathsf{ACA}_0$ (see Corollary 3.4.7) every extension of a scattered FAC partial order is scattered, it suffices to show that $\sum_{\sigma\in T} P_\sigma$ is scattered. Also, $\mathsf{RCA}_0$ shows that every sum of scattered partial orders along a scattered partial order is scattered (see Lemma 1.5.2). Therefore it is sufficient to show that $E(T)$ is scattered. Suppose not and let $D \subseteq E(T)$ be a dense chain. Let $\tau \in T$ be minimal such that $\tau$ has two incomparable extensions $\sigma_0, \sigma_1 \in D$. Fix $\sigma_0, \sigma_1 \in D$ such that $\sigma_0 < \sigma_1$ and $\tau \subseteq \sigma_i$ for all $i < 2$. Now let $x_0, x_1 \in P_\tau$ such that $\tau^\frown\langle x_i\rangle \subseteq \sigma_i$ for $i < 2$. It is not difficult to show that

$$D' = \{x \in P_\tau \colon x_0 <_{P_\tau} x <_{P_\tau} x_1 \wedge (\exists \sigma \in A)\tau^\frown\langle x\rangle \subseteq \sigma\}$$

is a dense chain on $P_\tau$, against the assumption about $\mathcal{B}$. $\square$

## 6.5 Sum decomposition for FAC partial orders

In this section we prove a sum decomposition theorem for FAC partial orders which will be used later on.

**Theorem 6.5.1** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *If $P$ is a FAC partial order, then there exist a cofinal (coinitial) restriction $B \subseteq P$ and a partition $\{P_x : x \in B\}$ such that $B$ is a wpo (reverse wpo), $x \in P_x \subseteq P(\preceq x)$ ($x \in P_x \subseteq P(\succeq x)$) for every $x \in B$ and $P$ extends $\sum_{x \in B} P_x$.*

We first prove a technical lemma. We need the following definition.

**Definition 6.5.2.** Let $\theta(n, X)$ be an arithmetical formula. Define $H_\theta(\alpha, f)$ to be the formula $f : \alpha \to \mathbb{N} \wedge (\forall \beta < \alpha)((\exists n)\theta(n, f[\beta]) \implies \theta(f(\beta), f[\beta]))$, where $f[\beta] = \{n \in \mathbb{N} : (\exists \gamma < \beta) f(\gamma) = n\}$.

**Lemma 6.5.3** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *Let $\theta(n, X)$ be an arithmetical formula. Then $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$ proves*

$$(\exists \alpha)(\exists f)(H_\theta(\alpha, f) \textit{ and } f \textit{ is not one-to-one}).$$

*Proof.*

**Claim** ($\mathsf{ACA}_0$)**.** Suppose $H_\theta(\alpha, f)$ and $H_\theta(\beta, g)$. If $h : \alpha \to \beta$ is a strong embedding, then $f = g \circ h$. In particular $f[\alpha] \subseteq g[\beta]$.

By arithmetical transfinite induction we prove $(\forall \gamma < \alpha) f(\gamma) = g \circ h(\gamma)$. Assume by induction $f \restriction \gamma = g \circ h \restriction \gamma$. We want to show $f(\gamma) = g \circ h(\gamma)$. By the assumption $H(\alpha, f) \wedge H(\beta, g)$, we get $\theta(f(\gamma), f[\gamma])$ and $\theta(g(h(\gamma)), g[h(\gamma)])$. By induction, $f[\gamma] = (g \circ h)[\gamma]$. Since $h$ is a strong embedding, $(g \circ h)[\gamma] = g[h(\gamma)]$. The thesis follows by uniqueness.

**Claim** ($\mathsf{ATR}_0$)**.** For every well-order $\alpha$ there exists $f$ such that $H_\theta(\alpha, f)$.

By arithmetical transfinite recursion on $\beta < \alpha$, let $f(\beta)$ be the $\omega$-least $x$ such that $\theta(x, f[\beta])$. If such $x$ does not exist, let $f(\beta) = 0$. This completes the proof of the claim.

By $\mathbf{\Sigma}^1_2$ comprehension, let $\Omega = \{n : (\exists \alpha)(\exists f)(H(\alpha, f) \wedge n \in f[\alpha])\}$. By $\mathbf{\Sigma}^1_2$-$\mathsf{AC}_0$, which is available in $\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$, there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of well-orders such that $(\forall n \in \Omega)(\exists f)(H(\alpha_n, f) \wedge n \in f[\alpha_n])$.

Let $\alpha = \sum_{n\in\omega} \alpha_n$ and $f$ so that $H(\alpha, f)$. We first claim $f[\alpha] = \Omega$. One direction follows by definition. Let $n \in \Omega$. Then $(\exists g)(H(\alpha_n, g) \wedge n \in g[\alpha_n])$. Since $\alpha_n \preceq \alpha$, by the claim, $g[\alpha_n] \subseteq f[\alpha]$, and hence $n \in f[\alpha]$.

Finally, let $\beta$ so that $\alpha \prec \beta$ and $g$ such that $H(\beta, g)$. We show that $g$ is not one-to-one. Let $h\colon \alpha \to \beta$ be a strong embedding. By the claim, $f = g \circ h$. Since $\alpha \prec \beta$, $f[\alpha] \subseteq g[\delta]$ for some $\delta < \beta$. Since $f[\alpha] = \Omega$, there exists $\gamma < \alpha$ such that $g(\delta) = f(\gamma)$. Since $f = g \circ h$, $g(\delta) = g(h(\gamma))$. Now $h(\gamma) < \delta$, and so $g$ is not one-to-one. This completes the proof. □

*Proof of Theorem 6.5.1.* Let $P$ be a FAC partial order. For $x \in P$ and $X \subseteq P$, we say that $x$ is *minimal over* $X$ if and only if

$$x \notin {\downarrow} X \wedge (\forall y \preceq x)(y \notin {\downarrow} X \implies (\forall z \in X)(z \preceq x \implies z \preceq y)).$$

**Claim.** For all $X \subseteq P$, if $X$ is not cofinal in $P$ then there exists $x \in P$ which is minimal over $X$.

Suppose $X$ is not cofinal but there are no minimal elements over $X$. Then we can define by arithmetical recursion a sequence $(x_n, z_n)$ of elements of $P^2$ such that for all $n$:

- $x_n \notin {\downarrow} X$ and $z_n \in X$;
- $x_n \succ x_{n+1}$;
- $z_n \prec x_n$ and $z_n \perp x_{n+1}$.

Since $X$ is well-founded and $P$ is FAC, $X$ is a well-partial order. Therefore, there exists $n < m$ such that $z_n \preceq z_m$. It follows that $z_n \preceq z_m \prec x_m \preceq x_{n+1}$, against $z_n \perp x_{n+1}$. This completes the proof of the claim.

Let $\theta(x, X)$ be the arithmetical formula which says that $x \in P$, $X \subseteq P$ and either $X$ is cofinal in $P$ or $x$ is minimal over $X$.

By Lemma 6.5.3, there exist a well-order $\alpha$ and a function $f\colon \alpha \to P$ so that $H(\alpha, f)$ and $f$ is not one-to-one. Let $\beta < \alpha$ be least such that $f[\beta]$ is cofinal. By the claim above, since $f$ is not one-to-one, such $\beta$ exists because otherwise $f(\beta)$ is minimal over $f[\beta]$ and so $f(\beta) \notin f[\beta]$ for all $\beta < \alpha$. As a consequence $f(\gamma)$ is minimal over $f[\gamma]$ for all $\gamma < \beta$.

Let $B = f[\beta]$ and $P_{f(\gamma)} = \{x \preceq f(\gamma)\colon x \npreceq f(\delta) \text{ for all } \delta < \gamma\}$. Clearly, $B$ is well-founded because $f(\delta) \prec f(\gamma)$ implies $\delta < \gamma$ for all $\delta < \gamma < \beta$. Also, $x \in P_x$ for all $x \in B$.

We next check that $P$ extends $\sum_{x \in B} P_x$. We must show that $x \in P_{f(\gamma)}$, $y \in P_{f(\delta)}$ and $f(\gamma) \prec f(\delta)$ imply $x \prec y$. Since $f(\gamma)$ is minimal over $f[\gamma]$, $\gamma < \delta$ and so $f(\gamma) \in f[\delta]$. Since $f(\delta)$ is minimal over $f[\delta]$, $f(\gamma) \prec y$. It follows that $x \preceq f(\gamma) \prec y$. $\square$

## 6.6 Hausdorff for scattered FAC partial orders

Let $\mathcal{B}$ be the class of wpo's and reverse wpo's. We aim to show the following:

**Theorem 6.6.1** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *If $P$ is scattered FAC then $P \in \mathcal{C}(\mathcal{B})$.*

We first prove a technical lemma.

**Lemma 6.6.2** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *Let $P$ be a FAC partial order. Then:*

(1) *$P \in \mathcal{C}(\mathcal{B})$ if and only if $P(\preceq x) \in \mathcal{C}(\mathcal{B})$ for every $x \in P$ if and only if $P(\succeq x) \in \mathcal{C}(\mathcal{B})$ for every $x \in P$;*

(2) *$P \in \mathcal{C}(\mathcal{B})$ if and only if $P(\preceq x) \in \mathcal{C}(\mathcal{B})$ or $P(\succeq x) \in \mathcal{C}(\mathcal{B})$ for every $x \in P$*

*Proof.* (1) The left to right direction follows from the fact that $\mathcal{C}(\mathcal{B})$ is closed under restrictions. Let us show that $P \in \mathcal{C}(\mathcal{B})$ if $P(\preceq x) \in \mathcal{C}(\mathcal{B})$ for every $x \in P$. Since $P$ is FAC, by Theorem 6.5.1, $P$ extends $\sum_{x \in B} P_x$, where $B \subseteq P$ is well-founded (and hence a wpo) and $P_x \subseteq P(\preceq x)$ for all $x \in B$. Since $\mathcal{C}(\mathcal{B})$ is closed under restrictions, each $P_x \in \mathcal{C}(\mathcal{B})$. As $\mathcal{C}(\mathcal{B})$ is closed under sums along wpo's, $\sum_{x \in B} P_x \in \mathcal{C}(\mathcal{B})$. Finally, $\mathcal{C}(\mathcal{B})$ is closed under extensions and hence $P \in \mathcal{C}(\mathcal{B})$.

(2) Consider the right to left direction. Assume that $P = P_0 \cup P_1$, where $P_0 = \{x \in P \colon P(\preceq x) \in \mathcal{C}(\mathcal{B})\}$ and $P_1 = \{x \in P \colon P(\succeq x) \in \mathcal{C}(\mathcal{B})\}$. Note that $P_0$ and $P_1$ are $\mathbf{\Sigma}^1_2$ definable. We must prove that $P \in \mathcal{C}(\mathcal{B})$. By $\mathbf{\Sigma}^1_2$ reduction (which is equivalent to $\mathbf{\Pi}^1_2$ separation and to $\mathbf{\Delta}^1_2$ comprehension), there exists $A \subseteq P_0$ such that $B = P \setminus A \subseteq P_1$. We claim that $A, B \in \mathcal{C}(\mathcal{B})$. For every $x \in A$, $A(\preceq x) \subseteq P(\preceq x)$ and hence $A(\preceq x) \in \mathcal{C}(\mathcal{B})$, because $\mathcal{C}(\mathcal{B})$ is closed under restrictions. It follows by (1) that $A \in \mathcal{C}(\mathcal{B})$. The case of $B$ is analogous. Since $P$ extends $A \oplus B$ and $\mathcal{C}(\mathcal{B})$ is closed under extensions and sums along finite antichains, $P \in \mathcal{C}(\mathcal{B})$. $\square$

*Proof of Theorem 6.6.1.* Let $P$ be a FAC partial order and suppose that $P \notin \mathcal{C}(\mathcal{B})$. We aim to show that $P$ is not scattered.

Add to $P$ a least element $x_0$ and a greatest element $x_1$. Let $D = \{\frac{n}{2^m} \colon 0 \leq n \leq 2^m \text{ and } m \in \mathbb{N}\}$ be the set of dyadic rationals in $\big[0, 1\big]$.

By recursion, we define an embedding $f$ of $D$ into $P$. Let $f(0) = x_0$ and $f(1) = x_1$. By hypothesis, $[x_0, x_1]_P \notin \mathcal{C}(\mathcal{B})$. Suppose we have defined $f(d)$ for every $d \in D$ with denominator $< 2^m$ and fix $d = \frac{n}{2^m}$ so that $f(d)$ has not been defined yet. In particular $n$ is odd. Let $d_0 = \frac{n-1}{2^m}$ and $d_1 = \frac{n+1}{2^m}$. By induction, assume that $[f(d_0), f(d_1)]_P \notin \mathcal{C}(\mathcal{B})$. By part (2) of Lemma 6.6.2, we can choose an element $f(d) \in [f(d_0), f(d_1)]_P$ such that $[f(d_0), f(d)]_P \notin \mathcal{C}(\mathcal{B})$ and $[f(d), f(d_1)]_P \notin \mathcal{C}(\mathcal{B})$. □

## 6.7 Hausdorff for FAC partial orders

Let $\mathcal{B}$ be the class of (countable) partial orders which are either wpo's, reverse wpo's or linear orders. We aim to show the following:

**Theorem 6.7.1** ($\mathbf{\Pi}^1_2$-$\mathsf{CA}_0$)**.** *If $P$ is FAC then $P \in \mathcal{C}(\mathcal{B})$.*

**Definition 6.7.2** ($\mathsf{ACA}_0$)**.** Let $P$ be a FAC partial order. We define $\mathcal{A}(P)$ to be the set of (codes of) finite antichains of $P$.

**Lemma 6.7.3** ($\mathsf{ACA}_0$)**.** *A partial order $P$ is FAC if and only $\mathcal{A}(P)$ ordered by reverse inclusion is well-founded (i.e. there are no infinite sequences $A_0 \subset A_1 \subset A_2 \subset \ldots$ of antichains of $P$).*

**Lemma 6.7.4** ($\mathsf{ACA}_0$)**.** *Let $P$ be a partial order and $x \in P$. Suppose that $P_0 = P(\perp x)$ and $P_1 = P \setminus P(\perp x)$ are both in $\mathcal{C}(\mathcal{B})$. Then $P \in \mathcal{C}(\mathcal{B})$.*

*Proof.* Clearly $P$ extends $P_0 \oplus P_1$, which belongs to $\mathcal{C}(\mathcal{B})$ since $\mathcal{C}(\mathcal{B})$ is closed under sums along finite antichains. □

*Proof of Theorem 6.7.1.* Let $P$ be a FAC partial order. By transfinite induction on $A \in \mathcal{A}(P)$ we prove $P(\perp A) \in \mathcal{C}(\mathcal{B})$. For $A = \emptyset$, $P(\perp A) = P$ and so $P \in \mathcal{C}(\mathcal{B})$.

Actually, for simplicity of notation, we shall assume $P(\perp x) \in \mathcal{C}(\mathcal{B})$ for every $x \in P$ and show that $P \in \mathcal{C}(\mathcal{B})$.

By $\mathbf{\Sigma}^1_2$ comprehension, define a binary relation $\sim$ on $P$ by letting $x \sim y$ if and only if $x \perp y$ or $(x, y)_P \in \mathcal{C}(\mathcal{B})$.

**Claim.** The relation $\sim$ is an equivalence relation.

Reflexivity and symmetry are immediate. Thus, we check transitivity. Suppose $x \sim y$ and $y \sim z$ with $x, y, z$ distinct. If $x \perp z$, $x \sim z$ by definition. So, we may assume $x \prec z$.

We let $I = (x,z)_P$ and prove $I \in \mathcal{C}(\mathcal{B})$. There are four cases:
Case 1: $y \in I$. As $I_0 = I(\perp y) = I \cap P(\perp y)$ and by induction hypothesis $P(\perp y) \in \mathcal{C}(\mathcal{B})$, $I_0 \in \mathcal{C}(\mathcal{B})$ because $\mathcal{C}(\mathcal{B})$ is closed under restrictions. On the other hand, $I_1 = I \setminus I_0 = (x,y)_P + \{y\} + (y,z)_P$ and so $I_1 \in \mathcal{C}(\mathcal{B})$ because $\mathcal{C}(\mathcal{B})$ is closed under sums along linear orders. By Lemma 6.7.4, $I \in \mathcal{C}(\mathcal{B})$.
Case 2: $x \prec y$ and $y \perp z$. We define $I_0 = (x,z)_P \cap (x,y)_P$ and $I_1 = (x,z)_P \setminus (x,y)_P$. Thus, $I$ extends $I_0 \oplus I_1$. Now, $I_0, I_1 \in \mathcal{C}(\mathcal{B})$ because $I_0 \subseteq (x,y)_P$, $I_1 \subseteq P(\perp y)$ and $\mathcal{C}(\mathcal{B})$ is closed under restrictions. Then $I \in \mathcal{C}(\mathcal{B})$.
Case 3: $x \perp y$ and $y \prec z$. This is like case 2.
Case 4: $x \perp y$ and $y \perp z$. The claim follows from $I \subseteq P(\perp y)$.

**Claim.** Each equivalence class is convex.

If $x \prec y \prec z$ and $x \sim z$, then clearly $x \sim y$ since $(x,y)_P$ is a restriction of $(x,z)_P$ and $\mathcal{C}(\mathcal{B})$ is closed under restrictions.

**Claim.** Each equivalence class is in $\mathcal{C}(\mathcal{B})$.

Let $C$ be an equivalence class and $z \in C$. By Lemma 6.5.3, since $P(\perp z) \in \mathcal{C}(\mathcal{B})$ by induction hypothesis, it suffices to show that $L = C_0 = \{x \in C \colon x \prec z\}$ and $C_1 = \{x \in C \colon z \prec x\}$ are both in $\mathcal{C}(\mathcal{B})$. We just argue that $C_0 \in \mathcal{C}(\mathcal{B})$, the argument for $C_1$ being symmetric.

Since $C_0$ is FAC, by Theorem 6.5.1 there exist a coinitial subset $B \subseteq C_0$ and a partition $\{C_{0,x} \colon x \in B\}$ of $C_0$ such that $B$ is a reverse wpo, $x \in C_{0,x} \subseteq C_0(\succeq x)$ and $C_0$ extends $\sum_{x \in B} C_{0,x}$.

Since $C$ is an equivalence class, $(x,z)_P \in \mathcal{C}(\mathcal{B})$ for all $x \in B$. Moreover, $C_{0,x}$ is a restriction of $\{x\} + (x,z)_P$ and hence belongs to $\mathcal{C}(\mathcal{B})$. It follows by the closure properties of $\mathcal{C}(\mathcal{B})$ that $C_0 \in \mathcal{C}(\mathcal{B})$.

**Claim.** If $C$ and $D$ are distinct equivalence classes then either $C \prec D$ or $D \prec C$.

Since incomparable elements are equivalent, every element of $C$ is comparable with every element of $D$. Suppose for a contradiction that $x \prec y \prec x'$ with $x, x' \in C$ and $y \in D$. Since equivalence classes are convex, $y \in C$, a contradiction.

Therefore, we can can write $P$ as the sum of its equivalence classes along a linear order. It follows by the closure properties of $\mathcal{C}(\mathcal{B})$ that $P \in \mathcal{C}(\mathcal{B})$. $\square$

# 7

# Linear extensions[1]

## 7.1 Introduction

We introduce the notion of $\tau$-like partial order, where $\tau$ is one of the linear order types $\omega$, $\omega^*$, $\omega + \omega^*$, and $\zeta$. For example, being $\omega$-like means that every element has finitely many predecessors, while being $\zeta$-like means that every interval is finite. We consider statements of the form "any $\tau$-like partial order has a $\tau$-like linear extension" and "any $\tau$-like partial order is embeddable into $\tau$". Working in the framework of reverse mathematics, we show that these statements are equivalent either to $\mathsf{B}\Sigma^0_2$ or to $\mathsf{ACA}_0$ over the usual base system $\mathsf{RCA}_0$.

Szpilrajn's Theorem ([Szp30]) states that any partial order has a linear extension. This theorem rises many natural questions, where in general we search for properties of the partial order which are preserved by some or all its linear extensions. For example it is well-known that a partial order is a well partial order if and only if all its linear extensions are well-orders.

A question which has been widely considered is the following: given a linear order type $\tau$, is it the case that any partial order, which does not embed $\tau$, can be extended to a linear order which does not embed $\tau$ either? If the answer is affirmative, $\tau$ is said to be extendible, while $\tau$ is weakly extendible if the same holds for any countable partial order. For instance, the order types of the natural numbers, of the integers, and of the rationals are extendible. Bonnet ([Bon69]) and Jullien ([Jul69]) characterized all countable extendible and weakly extendible linear order types respectively.

We are interested in a similar question: given a linear order type $\tau$ and a property characterizing $\tau$ and its suborders, is it true that any partial order which satisfies that property has a linear extension which also satisfies the same property? In our terminology:

[1] The content of this chapter also appears in [FM12]

does any $\tau$-like partial order have a $\tau$-like linear extension? Here we address this question for the linear order types $\omega$, $\omega^*$ (the inverse of $\omega$), $\omega + \omega^*$ and $\zeta$ (the order of integers). So, from now on, $\tau$ will denote one of these.

**Definition 7.1.1.** Let $(P, \leq_P)$ be a countable partial order. We say that $P$ is

- $\omega$-*like* if every element of $P$ has finitely many predecessors;
- $\omega^*$-*like* if every element of $P$ has finitely many successors;
- $\omega + \omega^*$-*like* if every element of $P$ has finitely many predecessors or finitely many successors;
- $\zeta$-*like* if for every pair of elements $x, y \in P$ there exist only finitely many elements $z$ with $x <_P z <_P y$.

The previous definition resembles Definition 2.3 of Hirschfeldt and Shore ([HS07]), where linear orders of type $\omega$, $\omega^*$ and $\omega + \omega^*$ are introduced. The main difference is that the order properties defined by Hirschfeldt and Shore are meant to uniquely determine a linear order type up to isomorphism, whereas our definitions apply to partial orders in general and do not determine an order type. Notice also that, for instance, an $\omega$-like partial order is also $\omega + \omega^*$-like and $\zeta$-like.

We introduce the following terminology:

**Definition 7.1.2.** We say that $\tau$ is *linearizable* if every $\tau$-like partial order has a linear extension which is also $\tau$-like.

With this definition in hand, we are ready to formulate the results we want to study:

**Theorem 7.1.3.** *The following hold:*

(1) $\omega$ *is linearizable;*

(2) $\omega^*$ *is linearizable;*

(3) $\omega + \omega^*$ *is linearizable;*

(4) $\zeta$ *is linearizable.*

A proof of the linearizability of $\omega$ can be found in Fraïssé's monograph ([Fra00, §2.15]), where the result is attributed to Milner and Pouzet. $(2)$ is similar to $(1)$ and the proof of $(3)$ easily follows from $(1)$ and $(2)$. The linearizability of $\zeta$ is apparently a new result (for a proof see Lemma 7.3.2 below).

In this chapter we study the statements contained in Theorem 7.1.3 from the standpoint of reverse mathematics (the standard reference is [Sim09]), whose goal is to characterize the axiomatic assumptions needed to prove mathematical theorems. We assume the reader is familiar with systems such as $\mathsf{RCA}_0$ and $\mathsf{ACA}_0$. The reverse mathematics of weak extendibility is studied in [DHLS03] and [Mon06]. The existence of maximal linear extensions of well partial orders is studied from the reverse mathematics viewpoint in [MS11].

Our main result is that the linearizability of $\tau$ is equivalent over $\mathsf{RCA}_0$ to the $\mathbf{\Sigma}^0_2$ bounding principle $\mathsf{B}\mathbf{\Sigma}^0_2$ when $\tau \in \{\omega, \omega^*, \zeta\}$, and to $\mathsf{ACA}_0$ when $\tau = \omega + \omega^*$. For more details on $\mathsf{B}\mathbf{\Sigma}^0_2$, including an apparently new equivalent (simply asserting that a finite union of finite sets is finite), see §7.2 below.

The linearizability of $\omega$ appears to be the first example of a genuine mathematical theorem (actually appearing in the literature for its own interest, and not for its metamathematical properties) that turns out to be equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$.

To round out our reverse mathematics analysis, we also consider a notion closely related to linearizability:

**Definition 7.1.4.** We say that $\tau$ is *embeddable* if every $\tau$-like partial order $P$ embeds into $\tau$, that is there exists an order preserving map from $P$ to $\tau$.[2]

It is rather obvious that $\tau$ is linearizable if and only if $\tau$ is embeddable. Let us notice that $\mathsf{RCA}_0$ easily proves that embeddable implies linearizable. Not surprisingly, the converse is not true. In fact, we show that embeddability is strictly stronger when $\tau \in \{\omega, \omega^*, \zeta\}$, and indeed equivalent to $\mathsf{ACA}_0$. The only exception is given by $\omega + \omega^*$, for which both properties are equivalent to $\mathsf{ACA}_0$.

We use the following definitions in $\mathsf{RCA}_0$.

**Definition 7.1.5** ($\mathsf{RCA}_0$)**.** Let $\leq$ denote the usual ordering of natural numbers. The linear order $\omega$ is $(\mathbb{N}, \leq)$, while $\omega^*$ is $(\mathbb{N}, \geq)$.

[2]To formalize this definition in $\mathsf{RCA}_0$, we need to fix a canonical representative of the order type $\tau$, which we do in Definition 1.5.

Let $\{P_i : i \in Q\}$ be a family of partial orders indexed by a partial order $Q$. The *lexicographic sum* of the $P_i$ along $Q$, denoted by $\sum_{i \in Q} P_i$, is the partial order on the set $\{(i,x) : i \in Q \wedge x \in P_i\}$ defined by

$$(i,x) \leq (j,y) \iff i <_Q j \vee (i = j \wedge x \leq_{P_i} y).$$

The *sum* $\sum_{i<n} P_i$ can be regarded as the lexicographic sum along the $n$-element chain. In particular $P_0 + P_1$ is the lexicographic sum along the 2-element chain (and we have thus defined $\omega + \omega^*$ and $\zeta = \omega^* + \omega$).

Similarly, the *disjoint sum* $\bigoplus_{i<n} P_i$ is the lexicographic sum along the $n$-element antichain.

## 7.2 $\Sigma^0_2$ bounding and finite union of finite sets

Let us recall that $\mathsf{B}\mathbf{\Sigma}^0_2$ (standing for $\mathbf{\Sigma}^0_2$ bounding, and also known as $\mathbf{\Sigma}^0_2$ collection) is the scheme:

$$(\forall i < n)(\exists m)\varphi(i,n,m) \implies (\exists k)(\forall i < n)(\exists m < k)\varphi(i,n,m), \qquad (\mathsf{B}\mathbf{\Sigma}^0_2)$$

where $\varphi$ is any $\mathbf{\Sigma}^0_2$ formula.

It is well-known that $\mathsf{RCA}_0$ does not prove $\mathsf{B}\mathbf{\Sigma}^0_2$, which is strictly weaker than $\mathbf{\Sigma}^0_2$ induction. Neither of $\mathsf{WKL}_0$ and $\mathsf{B}\mathbf{\Sigma}^0_2$ implies the other and Hirst ([Hir87], for an accessible proof see [CJS01, Theorem 2.11]) showed that $\mathsf{RT}^2_2$ (Ramsey theorem for pairs and two colors) implies $\mathsf{B}\mathbf{\Sigma}^0_2$.

A few combinatorial principles are known to be equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$ over $\mathsf{RCA}_0$.

Hirst ([Hir87], for an accessible proof see [CJS01, Theorem 2.10]) showed that, over $\mathsf{RCA}_0$, $\mathsf{B}\mathbf{\Sigma}^0_2$ is equivalent to the infinite pigeonhole principle, i.e. the statement

$$(\forall n)(\forall f : \mathbb{N} \to n)(\exists A \subseteq \mathbb{N} \text{ infinite})(\exists c < n)(\forall m \in A)(f(m) = c). \qquad (\mathsf{RT}^1_{<\infty})$$

(The notation arises from viewing the infinite pigeonhole principle as Ramsey theorem for singletons and an arbitrary finite number of colors.)

Chong, Lempp and Yang ([CLY10]) showed that a combinatorial principle $\mathsf{PART}$ about infinite $\omega + \omega^*$ linear orders, introduced by Hirschfeldt and Shore ([HS07, §4]), is also equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$. More recently, Hirst ([Hir12]) also proved that $\mathsf{B}\mathbf{\Sigma}^0_2$ is equivalent to a statement apparently similar to Hindman's theorem, but much weaker from the reverse mathematics viewpoint.

We consider the statement that a finite union of finite sets is finite:

$$(\forall i < n)(X_i \text{ is finite}) \implies \bigcup_{i<n} X_i \text{ is finite.} \qquad (\mathsf{FUF})$$

Here "$X$ is finite" means $(\exists m)(\forall x \in X)(x < m)$. This statement can be viewed as a second-order version of $\Pi_0$ regularity, which in the context of first-order arithmetic is known to be equivalent to $\Sigma_2$ bounding (see e.g. [HP93, Theorem 2.23.4]).

**Lemma 7.2.1.** *Over* $\mathsf{RCA}_0$, $\mathsf{B}\boldsymbol{\Sigma}^0_2$ *is equivalent to* $\mathsf{FUF}$.

*Proof.* First notice that $\mathsf{FUF}$ follows immediately from the instance of $\mathsf{B}\boldsymbol{\Sigma}^0_2$ relative to the $\boldsymbol{\Pi}^0_1$, and hence $\boldsymbol{\Sigma}^0_2$, formula $(\forall x \in X_i)(x < m)$.

For the other direction we use Hirst's result recalled above: it suffices to prove that $\mathsf{FUF}$ implies $\mathsf{RT}^1_{<\infty}$. Let $f\colon \mathbb{N} \to n$ be given. Define for each $i < n$ the set $X_i = \{m\colon f(m) = i\}$. Clearly $\bigcup_{i<n} X_i = \mathbb{N}$ is infinite. By $\mathsf{FUF}$, there exists $i < n$ such that $X_i$ is infinite. Now $X_i$ is an infinite homogeneous set for $f$. □

## 7.3 Equivalences with $\mathsf{B}\boldsymbol{\Sigma}^0_2$

Notice that Szpilrajn's Theorem is easily seen to be computably true (see [Dow98, Observation 6.1]) and provable in $\mathsf{RCA}_0$. We use this fact several times without further notice.

We start by proving that $\mathsf{B}\boldsymbol{\Sigma}^0_2$ suffices to establish the linearizability of $\omega$, $\omega^*$ and $\zeta$.

**Lemma 7.3.1.** $\mathsf{RCA}_0$ *proves that* $\mathsf{B}\boldsymbol{\Sigma}^0_2$ *implies the linearizability of* $\omega$ *and* $\omega^*$.

*Proof.* We argue in $\mathsf{RCA}_0$ and, by Lemma 7.2.1, we may assume $\mathsf{FUF}$. Let us consider first $\omega$. So let $P$ be an $\omega$-like partial order which, to avoid trivialities, we may assume to be infinite. We recursively define a sequence $z_n \in P$ by letting $z_n$ be the least (w.r.t. the usual ordering of $\mathbb{N}$) $x \in P$ such that $(\forall i < n)(x \nleq_P z_i)$.

We show by $\boldsymbol{\Sigma}^0_1$ induction that $z_n$ is defined for all $n \in \mathbb{N}$. Suppose that $z_i$ is defined for all $i < n$. We want to prove $(\exists x \in P)(\forall i < n)(x \nleq_P z_i)$. Define $X_i = \{x \in P\colon x \leq_P z_i\}$ for $i < n$. Since $P$ is $\omega$-like, each $X_i$ is finite. By $\mathsf{FUF}$, $\bigcup_{i<n} X_i$ is also finite. The claim follows from the fact that $P$ is infinite.

Now define for each $n \in \mathbb{N}$ the finite set

$$P_n = \{x \in P \colon x \leq_P z_n \wedge (\forall i < n)(x \nleq_P z_i)\}.$$

It is not hard to see that the $P_n$'s form a partition of $P$, and that if $x \leq_P y$ with $x \in P_i$ and $y \in P_j$, then $i \leq j$. Then let $L$ be a linear extension of the lexicographic sum $\sum_{n \in \omega} P_n$. $L$ is clearly a linear order and extends $P$ by the remark above. To prove that $L$ is $\omega$-like, note that the set of $L$-predecessors of an element of $P_n$ is included in $\bigcup_{i \leq n} P_i$, which is finite, by $\mathsf{FUF}$ again.

For $\omega^*$, repeat the same construction using $\geq_P$ in place of $\leq_P$, and let $L$ be a linear extension of $\sum_{n \in \omega^*} P_n$. $\square$

**Lemma 7.3.2.** $\mathsf{RCA}_0$ *proves that* $\mathsf{B}\mathbf{\Sigma}^0_2$ *implies the linearizability of* $\zeta$.

*Proof.* In $\mathsf{RCA}_0$ assume $\mathsf{FUF}$. Let $P$ be a $\zeta$-like partial order, which we may again assume to be infinite. It is convenient to use the notation $[x, y]_P = \{z \in P \colon x \leq_P z \leq_P y \vee y \leq_P z \leq_P x\}$, so that $[x, y]_P \neq \emptyset$ whenever $x$ and $y$ are comparable.

We define by recursion a sequence $z_n \in P$ by letting $z_n$ be the least (w.r.t. the ordering of $\mathbb{N}$) $x \in P$ such that

$$x \notin \bigcup_{i,j<n} [z_i, z_j]_P.$$

As before, since $P$ is infinite and $\zeta$-like, one can prove using $\mathbf{\Sigma}^0_1$ induction and $\mathsf{FUF}$ that $z_n$ is defined for every $n \in \mathbb{N}$. It is also easy to prove that

$$P = \bigcup_{i,j \in \mathbb{N}} [z_i, z_j]_P.$$

Define for each $n \in \mathbb{N}$ the set

$$P_n = \bigcup_{i<n} [z_i, z_n]_P \setminus \bigcup_{i,j<n} [z_i, z_j]_P.$$

By $\mathsf{FUF}$, the $P_n$'s are finite. Moreover, they clearly form a partition of $P$. Note also that $z_n \in P_n$ and every element of $P_n$ is comparable with $z_n$. Furthermore, every interval $[x, y]_P$ is included in some $[z_i, z_j]_P$. Notice that the same holds for any partial order extending $\leq_P$.

We now extend $\leq_P$ to a partial order $\preceq_P$ such that any linear extension of $(P, \preceq_P)$ is $\zeta$-like. We say that $n$ is left if $z_n \leq_P z_i$ for some $i < n$; otherwise, we say that $n$ is right.

Notice that, since $z_n \in P_n$, $n$ is right if and only if $z_i \leq_P z_n$ for some $i < n$ or $z_n$ is incomparable with every $z_i$ with $i < n$.

The order $\preceq_P$ places $P_n$ below or above every $P_i$ with $i < n$ depending on whether $n$ is left or right. Formally, for $x, y \in P$ such that $x \in P_n$ and $y \in P_m$ let

$$x \preceq_P y \iff (n = m \wedge x \leq_P y) \vee (n < m \wedge m \text{ is right}) \vee (m < n \wedge n \text{ is left}).$$

We claim that $\preceq_P$ extends $\leq_P$. Let $x \leq_P y$ with $x \in P_n$ and $y \in P_m$. If $n = m$, $x \preceq_P y$ by definition. Suppose now that $n < m$, so that we need to prove that $m$ is right. As $x \in P_n$, $z_i \leq_P x$ for some $i \leq n$. Since $y \in P_m$, $y$ is comparable with $z_m$. Suppose that $z_m <_P y$. Then $y \leq_P z_j$ for some $j < m$, and so $z_i \leq_P x \leq_P y \leq_P z_j$ with $i, j < m$, contrary to $y \in P_m$. It follows that $y \leq_P z_m$ and thereby $z_i \leq_P z_m$ with $i < m$. Therefore, $m$ is right, as desired. The case $n > m$ (where we need to prove that $n$ is left) is similar.

We claim that $(P, \preceq_P)$ is still $\zeta$-like. To see this, it is enough to show that for all $i, j < n$

$$\{x \in P \colon z_i \preceq_P x \preceq_P z_j\} \subseteq \bigcup_{k<n} P_k$$

and apply $\mathsf{FUF}$. Let $x \in P_k$ be such that $z_i \prec_P x \prec_P z_j$. Suppose, for a contradiction, that $k \geq n$ and hence that $i, j < k$. By the definition of $\preceq_P$, $z_i \prec_P x$ implies that $k$ is right. At the same time, $x \prec_P z_j$ implies that $k$ is left, a contradiction.

Now let $L$ be any linear extension of $(P, \preceq_P)$ and hence of $(P, \leq_P)$. We claim that $L$ is $\zeta$-like. To prove this, we show that for all $i, j \in \mathbb{N}$

$$\{x \in P \colon z_i \leq_L x \leq_L z_j\} = \{x \in P \colon z_i \preceq_P x \preceq_P z_j\}.$$

One inclusion is obvious because $\leq_L$ extends $\preceq_P$. For the converse, observe that the $z_n$'s are $\preceq_P$-comparable with any other element. $\square$

We can now state and prove our reverse mathematics results.

**Theorem 7.3.3.** *Over* $\mathsf{RCA}_0$, *the following are pairwise equivalent:*

(1) $\mathsf{B}\mathbf{\Sigma}^0_2$;

(2) $\omega$ *is linearizable;*

(3) $\omega^*$ *is linearizable;*

(4) *$\zeta$ is linearizable.*

*Proof.* Lemma 7.3.1 gives $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. The implication $(1) \Rightarrow (4)$ is Lemma 7.3.2.

To show $(2) \Rightarrow (1)$, we assume linearizability of $\omega$ and prove $\mathsf{FUF}$. So let $\{X_i : i < n\}$ be a finite family of finite sets. We define $P = \bigoplus_{i<n}(X_i + \{m_i\})$, where the $m_i$'s are distinct and every $X_i$ is regarded as an antichain. $P$ is $\omega$-like, and so by $(2)$ there exists an $\omega$-like linear extension $L$ of $P$. Let $m_j$ be the $L$-maximum of $\{m_i : i < n\}$. Then $\bigcup_{i<n} X_i$ is included in the set of $L$-predecessors of $m_j$, and is therefore finite because $L$ is $\omega$-like.

The implication $(3) \Rightarrow (1)$ is analogous. For $(4) \Rightarrow (1)$, prove $\mathsf{FUF}$ by using the partial order $\bigoplus_{i<n}(\{\ell_i\} + X_i + \{m_i\})$. □

We now show that the linearizability of $\omega + \omega^*$ requires $\mathsf{ACA}_0$.

**Theorem 7.3.4.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

1. $\mathsf{ACA}_0$*;*
2. $\omega + \omega^*$ *is linearizable.*

*Proof.* We begin by proving $(1) \Rightarrow (2)$. Let $P$ be an $\omega+\omega^*$-like partial order. In $\mathsf{ACA}_0$ we can define the set $P_0$ of the elements having finitely many predecessors. So $P_1 = P \setminus P_0$ consists of elements having finitely many successors. Clearly, $P_0$ is $\omega$-like and $P_1$ is $\omega^*$-like. Since $\mathsf{ACA}_0$ is strong enough to prove $\mathsf{B}\Sigma^0_2$, by Lemma 7.3.1, $P_0$ has an $\omega$-like linear extension $L_0$ and $P_1$ has an $\omega^*$-like linear extension $L_1$. Since $P_0$ is downward closed and $P_1$ is upward closed, it is not difficult to check that the linear order $L = L_0 + L_1$ is $\omega + \omega^*$-like and extends $P$.

For the converse, let $f \colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. We set out to define an $\omega + \omega^*$-like partial order $P$ such that any $\omega + \omega^*$-like linear extension of $P$ encodes the range of $f$. To this end, we use an $\omega + \omega^*$-like linear order $A = \{a_n : n \in \mathbb{N}\}$ given by the false and true stages of $f$. Recall that $n \in \mathbb{N}$ is said to be true (for $f$) if $(\forall m > n)(f(m) > f(n))$ and false otherwise, and note that the range of $f$ is $\mathbf{\Delta}^0_1$ definable from any infinite set of true stages.

The idea for $A$ comes from the well-known construction of a computable linear order such that any infinite descending sequence computes $\emptyset'$. This construction can be carried out in $\mathsf{RCA}_0$ (see [MS11, Lemma 4.2]). Here, we define $A$ by letting $a_n \leq a_m$ if and only if either

$$f(k) < f(n) \text{ for some } n < k \leq m, \text{ or}$$
$$m \leq n \text{ and } f(k) > f(m) \text{ for all } m < k \leq n.$$

It is not hard to see that $A$ is a linear order. Moreover, if $n$ is false, then $a_n$ has finitely many predecessors and infinitely many successors. Similarly, if $n$ is true, then $a_n$ has finitely many successors and infinitely many predecessors. In particular, $A$ is an $\omega + \omega^*$-like linear order.

Now let $P = A \oplus B$ where $B = \{b_n \colon n \in \mathbb{N}\}$ is a linear order of order type $\omega^*$, defined by letting $b_n \leq b_m$ if and only if $n \geq m$. It is clear that $P$ is an $\omega + \omega^*$-like partial order. By hypothesis, there exists an $\omega + \omega^*$-like linear extension $L$ of $P$. We claim that $n$ is a false stage if and only if it satisfies the $\mathbf{\Pi}^0_1$ formula $(\forall m)(a_n <_L b_m)$.

In fact, if $n$ is false and $b_m \leq_L a_n$, then $b_m$ has infinitely many successors in $L$, since $a_n$ has infinitely many successors in $P$ and a fortiori in $L$. On the other hand, $b_m$ has infinitely many predecessors in $P$, and hence also in $L$, contradiction. Likewise, if $n$ is true and $a_n <_L b_m$ for all $m$, then $a_n$ has infinitely many successors as well as infinitely many predecessors in $L$, which is a contradiction again.

Therefore, the set of false stages is $\mathbf{\Delta}^0_1$, and so is the set of true stages, which thus exists in $\mathsf{RCA}_0$. This completes the proof. □

## 7.4 Equivalences with $\mathsf{ACA}_0$

We turn our attention to embeddability. As noted before, $\mathsf{RCA}_0$ suffices to prove that "$\tau$ is embeddable" implies "$\tau$ is linearizable". The converse is true in $\mathsf{ACA}_0$. Actually, embeddability is equivalent to $\mathsf{ACA}_0$. We thus prove the following.

**Theorem 7.4.1.** *The following are pairwise equivalent over* $\mathsf{RCA}_0$*:*

(1) $\mathsf{ACA}_0$;

(2) $\omega$ *is embeddable;*

(3) $\omega^*$ *is embeddable;*

(4) $\zeta$ *is embeddable;*

(5) $\omega + \omega^*$ *is embeddable;*

*Proof.* We first show that $(1)$ implies the other statements. Since $\mathsf{B}\boldsymbol{\Sigma}^0_2$ is provable in $\mathsf{ACA}_0$, it follows from Theorem 7.3.3 that $\mathsf{ACA}_0$ proves the linearizability of $\omega$, $\omega^*$ and $\zeta$. By Theorem 7.3.4, $\mathsf{ACA}_0$ proves the linearizability of $\omega + \omega^*$. We now claim that in $\mathsf{ACA}_0$ "$\tau$ is linearizable" implies "$\tau$ is embeddable" for each $\tau$ we are considering. The key fact is that the property of having finitely many predecessors (successors) in a partial order, as well as having exactly $n \in \mathbb{N}$ predecessors (successors), is arithmetical. Analogously, for a set, and hence for an interval, being finite or having size exactly $n \in \mathbb{N}$ is arithmetical too. (All these properties are in fact $\boldsymbol{\Sigma}^0_2$.)

We consider explicitly the case of $\omega + \omega^*$ (the other cases are similar). So let $L$ be a $\omega + \omega^*$-like linear extension of a given $\omega + \omega^*$-like partial order. We want to show that $L$ is embeddable into $\omega + \omega^*$. Define $f\colon L \to \omega + \omega^*$ by

$$f(x) = \begin{cases} (0, |\{y \in L\colon y <_L x\}|) & \text{if } x \text{ has finitely many predecessors,} \\ (1, |\{y \in L\colon x <_L y\}|) & \text{otherwise.} \end{cases}$$

It is easy to see that $f$ preserves the order.

For the reversals, notice that $(5) \Rightarrow (1)$ immediately follows from Theorem 7.3.4.

As the others are quite similar, we only see $(2) \Rightarrow (1)$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a given one-to-one function. We want to prove that the range of $f$ exists. We fix an antichain $A = \{a_m\colon m \in \mathbb{N}\}$ and elements $b^n_j$ for $n \in \mathbb{N}$ and $j \leq n$. The partial order $P$ is obtained by putting for each $n \in \mathbb{N}$ the $n+1$ elements $b^n_j$ below $a_{f(n)}$. Formally, $b^n_j \leq_P a_m$ when $f(n) \leq m$, and there are no other comparabilities.

$P$ is clearly an $\omega$-like partial order. Apply the hypothesis and obtain an embedding $h\colon P \to \omega$. Now, we claim that $m$ belongs to the range of $f$ if and only if $(\exists n < h(a_m))(f(n) = m)$. One implication is trivial. For the other, suppose that $f(n) = m$. By construction, $a_m$ has at least $n+1$ predecessors in $P$, and thus it must be $h(a_m) > n$. $\square$

# A

## A.1 Erdös-Rado

The following diagram summarizes our state of knowledge about $\mathsf{ER}^2_2$ . We do not know whether the implications are strict.

$$
\begin{tikzcd}
& \mathsf{ER}^2_2 \arrow[dl] \arrow[dr] & & & \\
\mathsf{RT}^2_2 \arrow[dr] & & \mathsf{st\text{-}ER}^2_3 \arrow[dl] \arrow[dr] \arrow[r, leftrightarrow] & \mathsf{wspo}(\mathsf{ant}) \to \mathsf{wspo}(\mathsf{set}) & \\
& \mathsf{st\text{-}RT}^2_2 \arrow[dr] & & \mathsf{st\text{-}ER}^2_2 \arrow[dl] \arrow[r, leftrightarrow] & \mathsf{wspo}(\mathsf{ant}) \to \mathsf{wspo}(\mathsf{ext}) \\
& & \mathsf{RT}^1_{<\infty} & &
\end{tikzcd}
$$

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